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LITCHFIELD PARK, ARIZONA

November 22, 1965 GERA-1085

THE DYNAMICS OF THE SEMICIRCULAR CANALS

Fernand Belanger

Electronics Engineering Division

with

SUMMARY AND INTRODUCTION

by

Robert Mayne
Manager
Advanced Systems and Technologies Div.

This report is in partial compliance to a contract under the

National Aeronautics and Space Administration
Manned Spacecraft Center
Gemini Flight Support Procurement Section
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SUMMARY

The paper investigates the theoretical response of the semicircular canals as defined by the differential equation proposed by Steinhausen.[16] Transfer functions are derived for the response to a steady-state sinusoidal input of displacement, velocity, and acceleration of the head. The response to various forms of transient inputs are then investigated, including sinusoidal, step, pulse, velocity and acceleration of the head. Responses to these various inputs were obtained by computer runs. The curves plotted from these data should simplify greatly computations of responses to the most common inputs of motion used by experimenters. The study of the response to a pulse input of velocity whereby the head is moved from one position to another led to unexpected results. The cupula overshoots the neutral position as the head is stopped, and the integrated velocity signal corresponds theoretically to a return of the head to the original position. The report discusses possible compensatory reactions to offset these erroneous sensory data.

INTRODUCTION

By
Robert Mayne

The response of the semicircular canals to various forms of inputs is defined by the differential equation proposed originally by Steinhausen (1933). The implications of this formulation have been studied by many investigators. Van Egmond, et al (1949),[2] showed that the equation was identical to that of an overdamped pendulum, and made a computation of the constants. Mayne[12] (1950) showed by a frequency analysis that the canals measure velocity within a given bandwidth of frequencies, a conclusion which has been supported by Jones, et al.[10] Niven and Hixson[8, 9, 15] used Laplace transformation to show close correlation between experimentally and theoretically determined phase shift of nystagmus with respect to steady-state frequency inputs. They derived frequency response curves for steady-state acceleration inputs. A previous report[11] discussed the evidence regarding the value of the constants in the differential equation and showed that these values have probably been grossly underestimated by van Egmond, et al. There does not appear to be in the literature a systematic determination of the theoretical responses of the semicircular canals to a wide variety of input motions. Mr. Belanger attempts to fill this gap in the present report.

The report was originally intended to serve as reference material and to assist in the computation of the responses to the most common types of motion inputs used by experimenters. It brought out, however, a point of considerable theoretical interest. In an angular movement of the head from one position to another, the velocity, starting from zero, must rise to a maximum then decrease to become zero again at the new position. It happens, however, that in an overdamped system as defined by Steinhausen's equation, the cupula overshoots the neutral position when the head comes to rest, and then executes a gradual exponential return to neutral. Mr. Belanger shows that if the crista should issue signals proportional to cupula displacement and that these signals are interpreted as a measure of velocity, their integration would correspond to a return of the head to the original position if the effects of threshold are neglected. The semicircular canal signals, by themselves, would be unsuitable, therefore, to measure angular displacement without compensatory reaction.

Mr. Belanger used Laplace transformations in the formal derivation of responses to various forms of inputs as the operational method is now the overwhelmingly accepted method of handling transient phenomena. It is hoped that the report may serve as a simplified introduction to the application of the technique to the analysis of transient vestibular reactions and body movements. The responses to various types of inputs were determined by computer runs and are presented in the report in the form of curves. These curves should make it convenient to compute responses for similar inputs of any magnitude.

It is well known that the response to the semicircular canals in the way of eye movements or sensations is affected by many factors and is by no means determined solely on the basis of computed cupula displacement.[1,3] Research should be oriented toward the discovery of the role played by every factor as it affects response. To this end the cupula movement and the corresponding vestibular signal should be properly appraised. It is hoped that the analysis presented in the report may be useful in this respect.

It is not enough, however, to compute properly cupula movement for a given input; it is also necessary to relate such computation to meaningful responses of the canal. It must be established what form of response out of the many which can be observed represents true unmodified semicircular canal output. Doubts have been expressed that this may be possible.[1] The findings in a previous report to the effect that sensations and nystagmus responses are in agreement with computations in the case of steady-state sinusoidal inputs are encouraging in indicating that they represent true canal behavior. The further hypothesis that the same condition obtains in willed movements, if confirmed, would make it possible to use responses as computed in the present report in the analysis of body movements. It would seem, also, that isolated canal response may be obtained by the recording of the firing rate of nerve fibers associated with the cristae when measured near the peripheral level in order to cut off possible efferent signals. Indications are that these responses agree with computations.[5]

Mr. Belanger shows how the overshoot of the cupula can be compensated by suitable operations performed on sensory data and that an integration of the signal could compensate completely for the effect of cupula stiffness to provide exact integration of acceleration. It is apparent, however, that such complete compensation is not effected as zero phase shift would be observed at all frequencies. Some partial integration may, however, take place. In any case, presently available data on the development of compensatory vestibular reactions to offset conflicting sensory data would lead to the conclusion that similar reactions would develop to harmonize visual and vestibular data. The problem requires further elucidation.

THE FREQUENCY RESPONSE OF THE SEMICIRCULAR CANALS

The dynamic behavior of the semicircular canals was first described by Steinhausen in the form of a differential equation relating the angular displacement $\theta_e(t)$ of the endolymph relative to the canal, to the angular displacement $\theta_i(t)$ impressed on the canal (or head). Others have investigated various canal characteristics on the basis of this equation, which can be written as,

$$\ddot{\theta}_e(t) + L\dot{\theta}_e(t) + P\theta_e(t) = \ddot{\theta}_i(t) \tag{1}$$

where

L is one canal constant in radians/second, and
P is another canal constant in radians/second^2 .

Using the Laplace transformation on Equation (1) gives the following (see Appendix).

$$s^2\theta_e(s) + Ls\theta_e(s) + P\theta_e(s) = s^2\theta_i(s) \; , \tag{2}$$

or

$$(s^2 + Ls + P)\theta_e(s) = s^2\theta_i(s) \tag{3}$$

where s is the Laplace operator.

From Equation (3) the transfer function relating $\theta_e(s)$ to $\theta_i(s)$ is obtained.

$$\frac{\theta_e(s)}{\theta_i(s)} = \frac{s^2}{s^2 + Ls + P} \; . \tag{4}$$

If it is desired to relate $\theta_e(s)$ to $\dot{\theta}_i(s)$, Equation (2) would be rewritten as,

$$s^2\theta_e(s) + Ls\theta_e(s) + P\theta_e(s) = s\,\dot{\theta}_i(s) \tag{5}$$

and, therefore, from Equation (5) would be obtained

$$\frac{\theta_e(s)}{\dot{\theta}_i(s)} = \frac{s}{s^2 + Ls + P} \; . \tag{6}$$

If it is desired to relate $\theta_e(s)$ to $\ddot{\theta}_i(s)$, Equation (2) would be written as,

$$s^2\theta_e(s) + Ls\,\theta_e(s) + P\theta_e(s) = \ddot{\theta}_i(s) \quad , \tag{7}$$

and, therefore, from Equation (7),

$$\frac{\theta_e(s)}{\ddot{\theta}_i(s)} = \frac{1}{s^2 + Ls + P} \quad . \tag{8}$$

Equations (4), (6), and (8) are transfer functions relating endolymph angular displacement relative to the canal to angular displacement, angular velocity, or angular acceleration, respectively, impressed on the canal (or head).

If the Laplace operator s is replaced by jω in Equations (4), (6), and (8), the right-hand members of those equations will become complex functions of the frequency ω (in radians/second).

Thus, each right-hand member can be modified to a form

$$Y(j\omega) = A(\omega)\, e^{jB(\omega)} \quad . \tag{9}$$

The function $A(\omega)$ is the amplitude frequency response while the function $B(\omega)$ is the phase frequency response.

Considering Equation (4) with $s = j\omega$,

$$\frac{\theta_e}{\theta_i}(j\omega) = Y_{\theta_i}(j\omega) = \frac{-\omega^2}{P - \omega^2 + j\,L\omega} = A_{\theta_i}(\omega)\, e^{jB_{\theta_i}(\omega)} \tag{10}$$

where

$$A_{\theta_i}(\omega) = \frac{\omega^2}{\left[(P - \omega^2)^2 + L^2\omega^2\right]^{\frac{1}{2}}} \tag{11}$$

and

$$B_{\theta_i}(\omega) = \pi - \tan^{-1}\left(\frac{L\,\omega}{P - \omega^2}\right) \quad . \tag{12}$$

Similarly, for Equations (6) and (8)

$$\frac{\theta_e}{\dot{\theta}_i}(j\omega) = Y_{\dot{\theta}_i}(j\omega) = \frac{j\omega}{P - \omega^2 + jL\omega} = A_{\dot{\theta}_i}(\omega)\, e^{jB_{\dot{\theta}_i}(\omega)} \tag{13}$$

where

$$A_{\dot{\theta}_i}(\omega) = \frac{\omega}{\left[(P - \omega^2)^2 + L^2\omega^2\right]^{\frac{1}{2}}} \tag{14}$$

and

$$B_{\dot{\theta}_i}(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{L\omega}{P - \omega^2}\right) \tag{15}$$

$$\frac{\theta_e}{\ddot{\theta}_i}(j\omega) = Y_{\ddot{\theta}_i}(j\omega) = \frac{1}{P - \omega^2 + jL\omega} = A_{\ddot{\theta}_i}(\omega)\, e^{jB_{\ddot{\theta}_i}(\omega)} \tag{16}$$

where

$$A_{\ddot{\theta}_i}(\omega) = \frac{1}{\left[(P - \omega^2)^2 + L^2\omega^2\right]^{\frac{1}{2}}} \tag{17}$$

and

$$B_{\ddot{\theta}_i}(\omega) = -\tan^{-1}\left(\frac{L\omega}{P - \omega^2}\right) \quad . \tag{18}$$

The semicircular canal dynamics have been compared to that of an overdamped torsional pendulum. This is indicative that the denominator function $(s^2 + Ls + P)$ of the right-hand member of Equations (4), (6), and (8) can be reduced to the product of two functions of s as follows.

$$s^2 + Ls + P = (s + \omega_1)(s + \omega_2) \tag{19}$$

where

$$P = \omega_1 \times \omega_2 = \omega_o^2 \tag{20}$$

and

$$L = \omega_1 + \omega_2 \quad . \tag{21}$$

The real constants ω_1 and ω_2 represent the lower and upper corner frequencies (in radians/second), respectively, for the various frequency response functions. The constant ω_o represents the natural frequency (in radians/second) of the canal.

In terms of ω_1 and ω_2, Equations (11), (14), and (17) can be re-written as

$$A_{\theta_i}(\omega) = \frac{\omega^2}{\left[(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)\right]^{\frac{1}{2}}} \tag{22}$$

$$A_{\dot{\theta}_i}(\omega) = \frac{\omega}{\left[(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)\right]^{\frac{1}{2}}} \tag{23}$$

$$A_{\ddot{\theta}_i}(\omega) = \frac{1}{\left[(\omega^2 + \omega_1^2)(\omega^2 + \omega_2^2)\right]^{\frac{1}{2}}} \, . \tag{24}$$

In terms of ω_1 and ω_2, the phase functions of Equations (12), (15), and (18) become

$$B_{\theta_i}(\omega) = \pi - \tan^{-1}\frac{\omega}{\omega_1} - \tan^{-1}\frac{\omega}{\omega_2} \tag{25}$$

$$B_{\dot{\theta}_i}(\omega) = \frac{\pi}{2} - \tan^{-1}\frac{\omega}{\omega_1} - \tan^{-1}\frac{\omega}{\omega_2} \tag{26}$$

$$B_{\ddot{\theta}_i}(\omega) = - \tan^{-1}\frac{\omega}{\omega_1} - \tan^{-1}\frac{\omega}{\omega_2} \, . \tag{27}$$

Thus, from Equation (1) originally developed by Steinhausen, three amplitude functions (Equations 22, 23, and 24) and three phase functions (Equations 25, 26, and 27) are developed, which completely describe the response of the canal in terms of angular displacement of endolymph relative to the canal, to canal (or head) angular displacement, velocity, and acceleration, respectively.

These frequency response functions are shown in Figures 1 and 2. The values assumed for L and P are 200.1 and 20, respectively, resulting in values of 200 and 0.1, respectively, for ω_2 and ω_1.

Considering the regions of flat response for the three frequency response curves of Figure 1, it may be argued that the semicircular canal behaves as an accelerometer with a scale factor of $\frac{1}{P}$ over the frequency range $0 \leq \omega \leq \omega_1$ (see the curve labeled $A_{\ddot{\theta}_i}(\omega)$).

Similarly, the curve labeled $A_{\dot{\theta}_i}(\omega)$ indicates that the semicircular canal behaves as a velocity meter with a scale factor of $\frac{1}{L}$ over the frequency range $\omega_1 \leq \omega \leq \omega_2$.

Finally, the curve labeled $A_{\theta_i}(\omega)$ indicates that the semicircular canal behaves as a position meter with a scale factor of 1 over the frequency range $\omega_2 \leq \omega \leq \infty$.

Since the frequency range of $\omega_1 \leq \omega \leq \omega_2$ would appear to cover quite well the range of frequencies associated with normal body motions, it may be assumed that the output from the canal would normally be interpreted by the central nervous system as being a measure of the head velocity. Thus, if the endolymph displacement angle θ_e is multiplied by the constant L, this would represent the <u>measured</u> head, or canal <u>velocity</u> $\dot{\theta}_m$.

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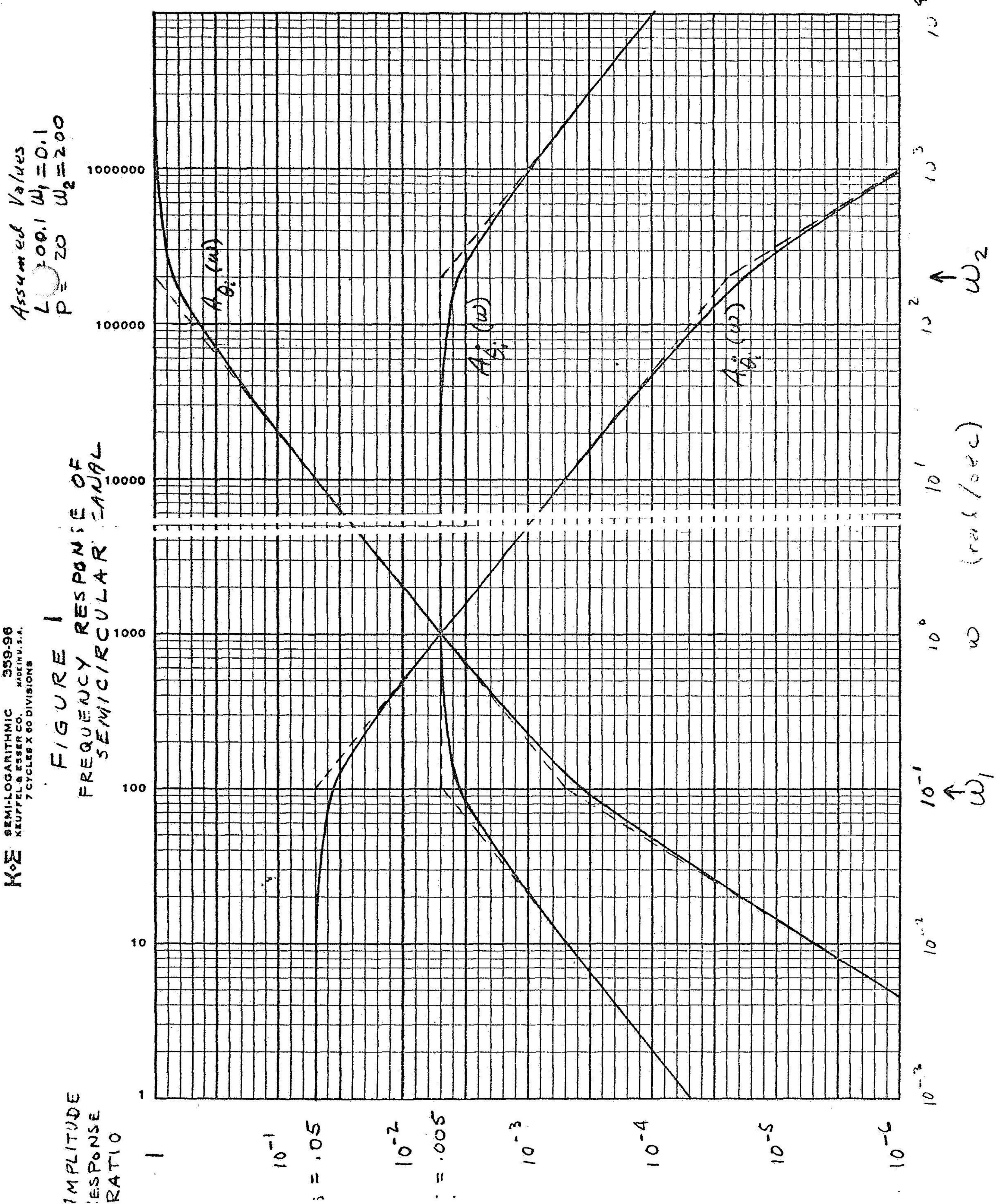

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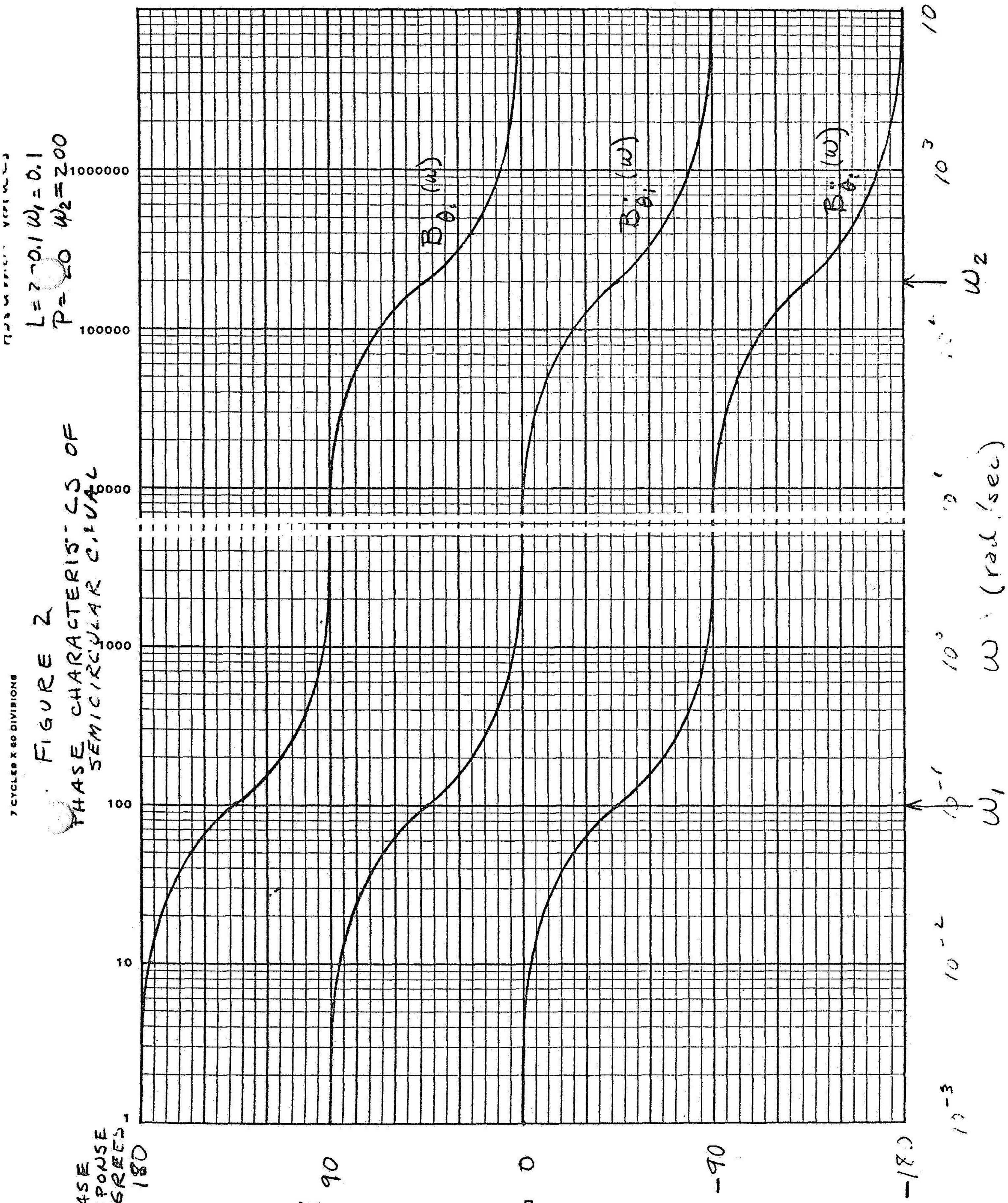

THE TRANSIENT RESPONSE OF THE SEMICIRCULAR CANALS

While the steady-state response of the semicircular canals to a sinusoidal input at a given frequency is completely defined in both amplitude and phase by the curve of Figures 1 and 2 (for the assumed values of L and P), the transient response of the canals to certain specific types of impressed head motions is of great interest to the experimenter.

The derivation of the transient responses of the canals will be based on the assumption of zero initial conditions in the state of the endolymph motion relative to the canal. Thus, at time t = 0, it is assumed that,

$$\theta_e(o) = 0 \tag{28}$$

$$\dot{\theta}_e(o) = 0 \tag{29}$$

$$\ddot{\theta}_e(o) = 0 \ . \tag{30}$$

1. <u>Canal Response to a Step Input of Head Velocity</u>
(Post-Rotational Experiment)

Since a velocity input is involved, Equation (6) will be used.

$$\theta_e(s) = \frac{s}{s^2 + Ls + P}\,\dot{\theta}_i(s) = \frac{s}{(s+\omega_1)(s+\omega_2)}\,\dot{\theta}_i(s) \tag{31}$$

Since a step input of velocity Ω is assumed, the function $\dot{\theta}_i(s) = \frac{\Omega}{s}$.

Thus,

$$\theta_e(s) = \frac{s}{(s+\omega_1)(s+\omega_2)}\,\frac{\Omega}{s} = \Omega\,\frac{1}{(s+\omega_1)(s+\omega_2)} \ . \tag{32}$$

If the function $\frac{1}{(s + \omega_1)(s + \omega_2)}$ is broken up by partial fraction expansion, Equation (32) can be re-written as (see Appendix),

$$\theta_e(s) = \Omega \left[\frac{1}{\omega_2 - \omega_1} \; \frac{1}{s + \omega_1} + \frac{1}{\omega_1 - \omega_2} \; \frac{1}{s + \omega_2} \right] \qquad (33)$$

or

$$\theta_e(s) = \Omega \frac{1}{\omega_2 - \omega_1} \left(\frac{1}{s + \omega_1} - \frac{1}{s + \omega_2} \right) . \qquad (34)$$

Taking the inverse Laplace transform of Equation (34) (see Appendix), the transient response $\theta_e(t)$ to the step input of velocity Ω becomes,

$$\theta_e(t) = \Omega \frac{1}{\omega_2 - \omega_1} \left(e^{-\omega_1 t} - e^{-\omega_2 t} \right) . \qquad (35)$$

The equivalent measured velocity is obtained from Equations(21) and (35) using the previous definition for measured velocity.

$$\dot{\theta}_m(t) = L\,\theta_e(t) = \Omega \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1} \left(e^{-\omega_1 t} - e^{-\omega_2 t} \right). \qquad (36)$$

If it is assumed that the central nervous system (CNS) is capable of integrating and differentiating the measured velocity signal, $\dot{\theta}_m(t)$, which it receives from the semicircular canal, for possible uses in eye movements or body control, these functions $\theta_m(t)$ and $\ddot{\theta}_m(t)$ are also of interest.

From Equation (36) and the assumption that $\theta_m(o) = 0$, is obtained,

$$\theta_m(t) = \Omega \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1} \left(- \frac{1}{\omega_1} e^{-\omega_1 t} + \frac{1}{\omega_2} e^{-\omega_2 t} + \frac{\omega_2 - \omega_1}{\omega_2 \omega_1} \right) \qquad (37)$$

and,

$$\ddot{\theta}_m(t) = \Omega \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(-\omega_1 e^{-\omega_1 t} + \omega_2 e^{-\omega_2 t}\right). \tag{38}$$

2. <u>Canal Response to a Step Input of Acceleration</u>

Because an acceleration input is specified, Equation (8) will be used.

$$\theta_e(s) = \frac{1}{s^2 + Ls + P}\,\ddot{\theta}_i(s) = \frac{1}{(s+\omega_1)(s+\omega_2)}\,\ddot{\theta}_i(s) . \tag{39}$$

With a step input of acceleration of magnitude A assumed, the input function $\ddot{\theta}_i(s)$ will be,

$$\ddot{\theta}_i(s) = \frac{A}{s} . \tag{40}$$

Thus,

$$\theta_e(s) = \frac{1}{(s+\omega_1)(s+\omega_2)}\,\frac{A}{s} = A\,\frac{1}{s(s+\omega_1)(s+\omega_2)} \tag{41}$$

By partial fraction expansion, Equation (41) becomes

$$\theta_e(s) = A\left[\frac{1}{\omega_2\omega_1}\,\frac{1}{s} - \frac{1}{\omega_1(\omega_2 - \omega_1)}\,\frac{1}{s+\omega_1} - \frac{1}{\omega_2(\omega_1 - \omega_2)}\,\frac{1}{s+\omega_2}\right] \tag{42}$$

or,

$$\theta_e(s) = A\,\frac{1}{\omega_2 - \omega_1}\left[-\frac{1}{\omega_1}\,\frac{1}{s+\omega_1} + \frac{1}{\omega_2}\,\frac{1}{s+\omega_2} + \frac{\omega_2 - \omega_1}{\omega_2\omega_1}\,\frac{1}{s}\right]. \tag{43}$$

Taking the inverse Laplace transform of Equation (43), the transient response $\theta_e(t)$ to the step input of acceleration A becomes,

$$\theta_e(t) = A\,\frac{1}{\omega_2 - \omega_1}\left(-\frac{1}{\omega_1} e^{-\omega_1 t} + \frac{1}{\omega_2} e^{-\omega_2 t} + \frac{\omega_2 - \omega_1}{\omega_2\,\omega_1}\right). \tag{44}$$

The <u>measured velocity</u> function, in this case will be

$$\dot{\theta}_m(t) = L\,\theta_e(t) = A\,\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(-\frac{1}{\omega_1} e^{-\omega_1 t} + \frac{1}{\omega_2} e^{-\omega_2 t} + \frac{\omega_2 - \omega_1}{\omega_2\omega_1}\right). \tag{45}$$

Again, differentiating and integrating Equation (45), the following is obtained.

$$\ddot{\theta}_m(t) = A\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(e^{-\omega_1 t} - e^{-\omega_2 t}\right) \tag{46}$$

and

$$\theta_m(t) = A\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(\frac{1}{\omega_1^2} e^{-\omega_1 t} - \frac{1}{\omega_2^2} e^{-\omega_2 t} - \frac{\omega_2^2 - \omega_1^2}{\omega_2^2\ \omega_1^2} + \frac{\omega_2 - \omega_1}{\omega_2\ \omega_1} t\right). \tag{47}$$

3. Summary of Response to Step Inputs of Velocity and Acceleration

In describing $\theta_m(t)$, $\dot{\theta}_m(t)$, and $\ddot{\theta}_m(t)$ for step inputs of velocity and acceleration, only four different functions have appeared. These are the following.

$$f_1(t) = \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(-\omega_1 e^{-\omega_1 t} + \omega_2 e^{-\omega_2 t}\right) \tag{48}$$

$$f_2(t) = \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(e^{-\omega_1 t} - e^{-\omega_2 t}\right) \tag{49}$$

$$f_3(t) = \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(-\frac{1}{\omega_1} e^{-\omega_1 t} + \frac{1}{\omega_2} e^{-\omega_2 t} + \frac{\omega_2 - \omega_1}{\omega_2\ \omega_1}\right) \tag{50}$$

and

$$f_4(t) = \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\left(\frac{1}{\omega_1^2} e^{-\omega_1 t} - \frac{1}{\omega_2^2} e^{-\omega_2 t} - \frac{\omega_2^2 - \omega_1^2}{\omega_2^2\ \omega_1^2} + \frac{\omega_2 - \omega_1}{\omega_2 \omega_1} t\right) \tag{51}$$

For a step input of velocity of magnitude Ω,

$$\ddot{\theta}_m(t) = \Omega f_1(t) \tag{52}$$

$$\dot{\theta}_m(t) = \Omega f_2(t) \tag{53}$$

$$\theta_m(t) = \Omega f_3(t) \ . \tag{54}$$

For a step input of acceleration of magnitude A,

$$\ddot{\theta}_m(t) = A\ f_2(t) \tag{55}$$

$$\dot{\theta}_m(t) = A\ f_3(t) \tag{56}$$

$$\theta_m(t) = A\ f_4(t)\ . \tag{57}$$

The functions $f_1(t)$, $f_2(t)$, $f_3(t)$, and $f_4(t)$ are presented in Figures 3, 4, 5, and 6, respectively.

Two curves are presented in each figure, corresponding to two sets of assumed values for ω_1 and ω_2 or L and P.

<u>Case 1</u> $\omega_1 = 0.1$ rad/sec L = 10.1
$\omega_2 = 10.0$ rad/sec P = 1.0

<u>Case 2</u> $\omega_1 = 0.1$ rad/sec L = 200.1
$\omega_2 = 200.0$ rad/sec P = 20.0 .

4. Canal Response to Pulse Inputs

The pulse input is characterized by its amplitude (Ω or A, depending on whether it is a pulse of velocity or acceleration), and its duration T. It may be thought of as being the result of two separate step inputs of the same amplitude (Ω or A), but of opposite sign, one occurring at t = 0 (and initiating the pulse), and the other occurring at t = T, and terminating the pulse by exactly cancelling the value of the first step function for $T < t$. The response of the semicircular canal to the first step input has already been covered in previous sections. Four functions have been developed describing varying responses to step inputs of velocity or acceleration.

One important characteristic of these functions is that they are zero (0) for $t < 0$. In other words, no response or output may exist prior to the start of the input function.

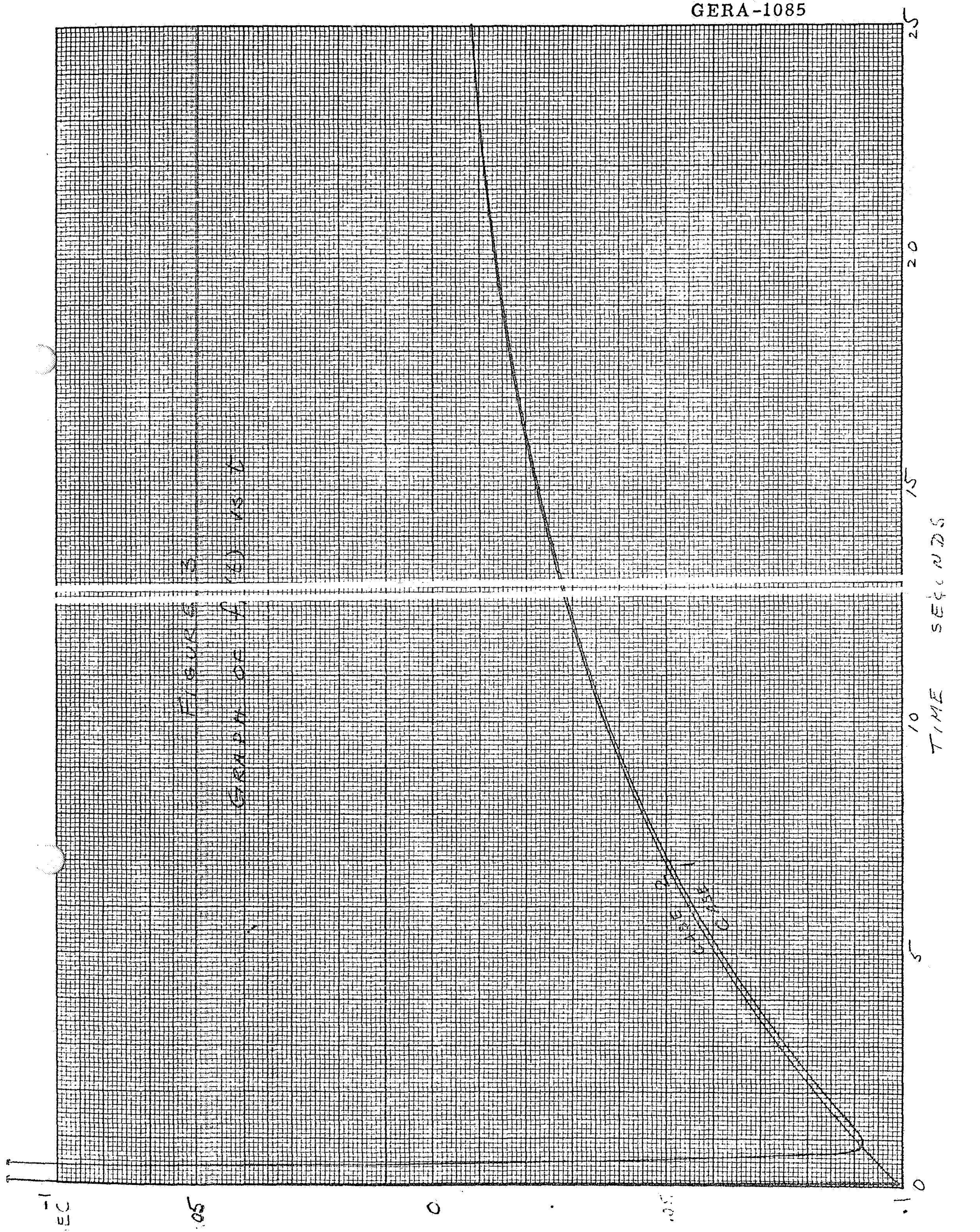
FIGURE 3
TIME SECONDS
0
5
10
15
20
25
CASE 2
CASE 1

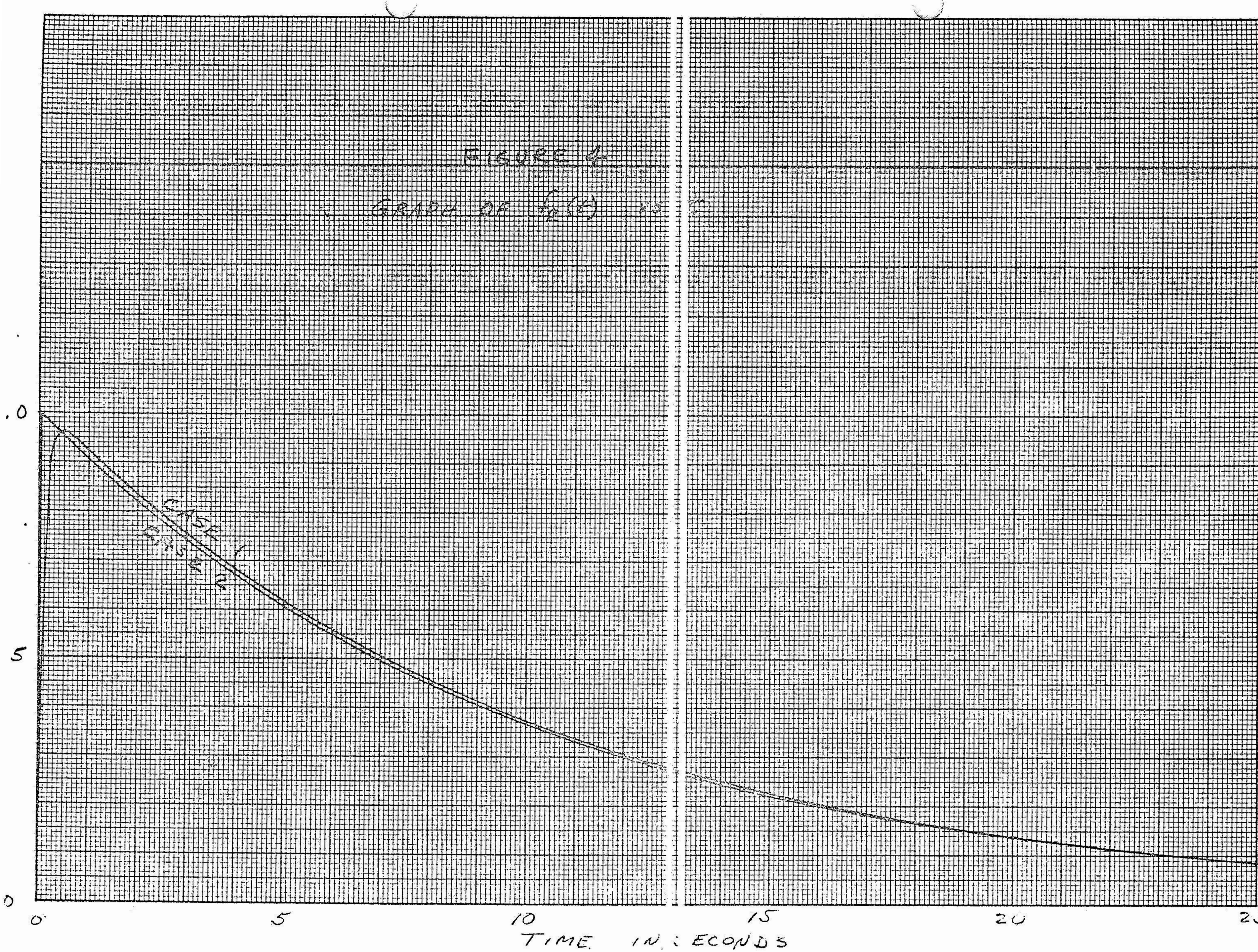
FIGURE 4
GRAPH OF $f_2(t)$
CASE 1
CASE 2
.0
5
0
0
5
10
15
20
25
TIME IN SECONDS

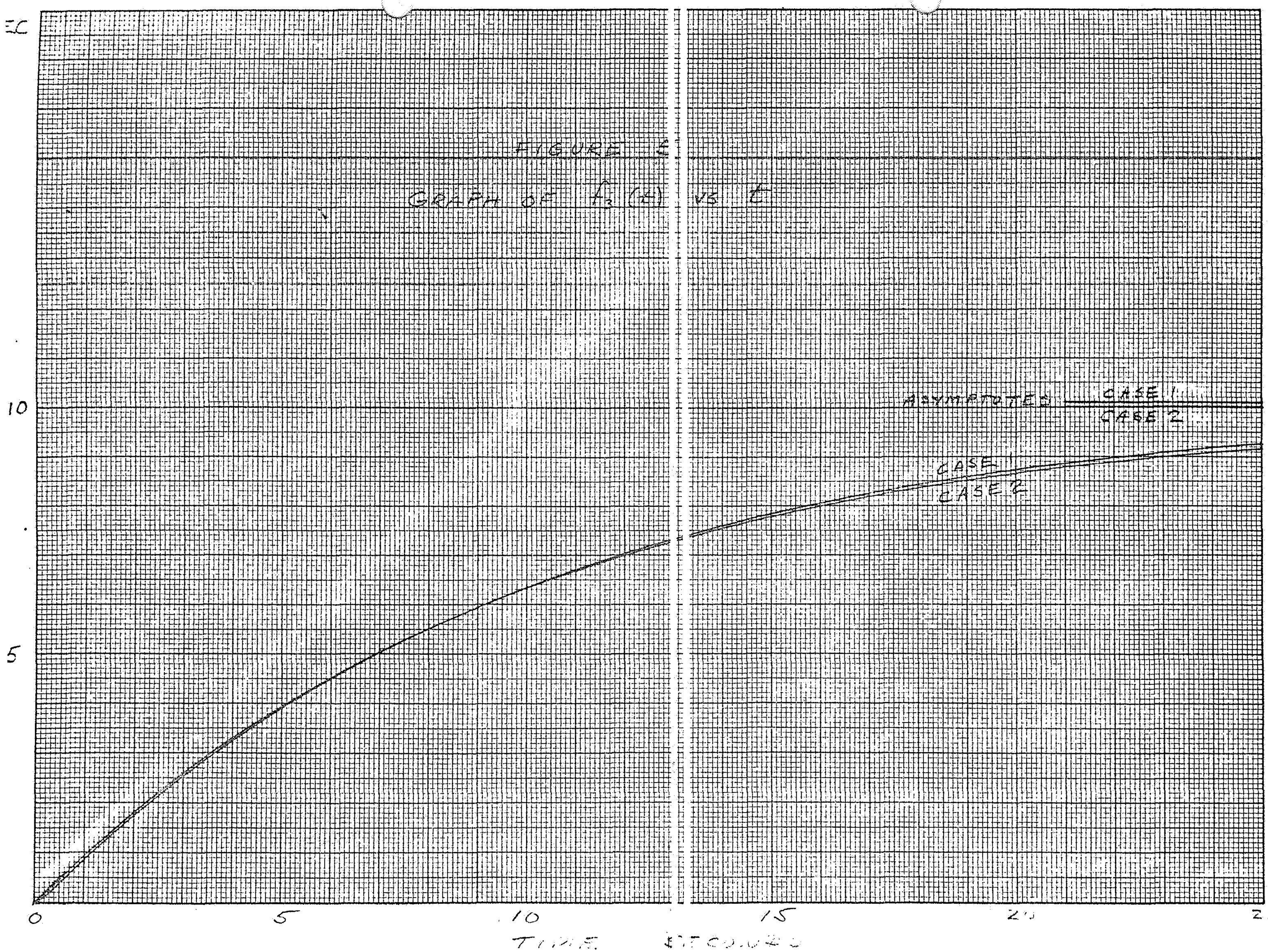
FIGURE 2
GRAPH OF $f_3(t)$ VS t
ASYMPTOTES
CASE 1
CASE 2
CASE 1
CASE 2
10
5
0
5
10
15
20
25
TIME

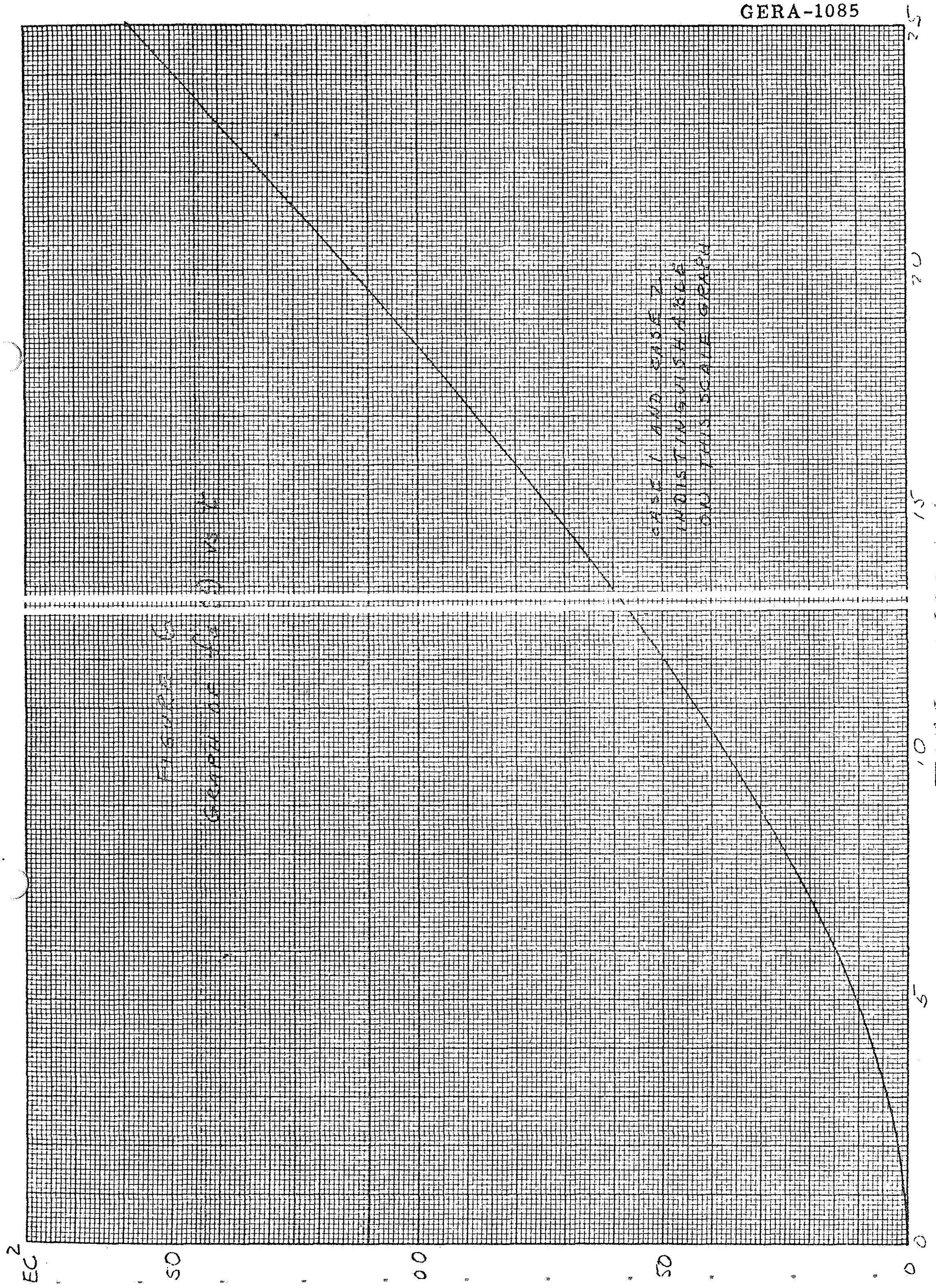

FIGURE 6
GRAPH OF
VS t
CASE 1 AND CASE 2
INDISTINGUISHABLE
ON THIS SCALE GRAPH
TIME SECONDS
0
5
10
15
20
25
EC2
50
00
50
0

The significance of this characteristic is that if the step input had been initiated at time $t = t_1$ instead of at $t = 0$, the response functions would have been written as $f(t - t_1)$ instead of $f(t)$, representing the original function $f(t)$ in shape, but displaced in time by an amount t_1, the time by which the step input was assumed to be delayed. Of course, in this case $f(t - t_1)$ would be zero for $t < t_1$.

Since the semicircular canal system response, and the assumed integration and differentiation processes in the CNS are all linear processes, the various response functions to a sum of input functions will be equal to the sum of the individual response functions to individual component functions of the input.

As an example, consider the function $\dot{\theta}_m(t)$ as described in Equation (53). This represents the semicircular canal response to a step of velocity Ω starting at $t = 0$. If an input step of the same magnitude had been initiated at time $t = T$, the response would then be

$$\dot{\theta}_m(t) = \Omega \, f_2(t - T) \quad . \tag{58}$$

Now, if the input is a pulse of velocity of Ω and duration T which may be considered the result of one step input of amplitude $+\Omega$ initiated at $t = 0$ and a second step input of amplitude $-\Omega$ initiated at $t = T$, the response $\dot{\theta}_m(t)$ in this case would be

$$\dot{\theta}_m(t) = \Omega \left[f_2(t) - f_2(t - T) \right] \quad . \tag{59}$$

Thus, the original function $f_2(t)$ shown in Figure 4 may be used in describing the canal response to a pulse of input velocity of given duration T.

This is shown in Figure 7 for $T = 5$ seconds. Similar considerations apply to the other response functions.

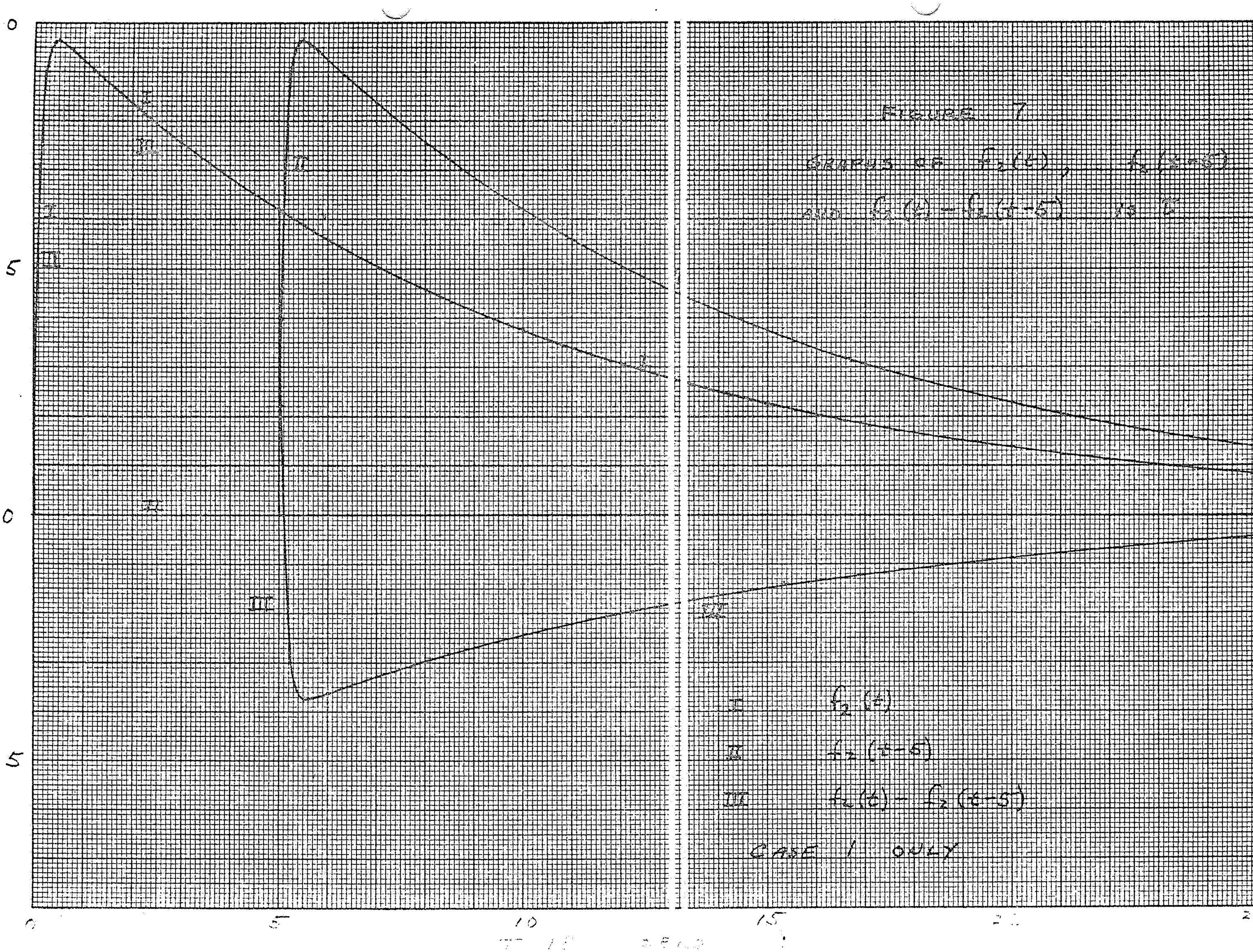
FIGURE 7
GRAPHS OF $f_2(t)$, $f_2(t-5)$
AND $f_2(t) - f_2(t-5)$ vs T
I $f_2(t)$
II $f_2(t-5)$
III $f_2(t) - f_2(t-5)$
CASE 1 ONLY
I
II
III
0
5
0
5
0
5
10
15
20
25

Thus, for a pulse input of velocity of magnitude Ω and duration T,

$$\ddot{\theta}_m(t) = \Omega \left[f_1(t) - f_1(t - T) \right], \tag{60}$$

and

$$\theta_m(t) = \Omega \left[f_3(t) - f_3(t - T) \right] \quad . \tag{61}$$

Similarly, for a pulse input of acceleration of magnitude A and duration T,

$$\ddot{\theta}_m(t) = A \left[f_2(t) - f_2(t - T) \right] \tag{62}$$

$$\dot{\theta}_m(t) = A \left[f_3(t) - f_3(t - T) \right] \tag{63}$$

$$\theta_m(t) = A \left[f_4(t) - f_4(t - T) \right] \quad . \tag{64}$$

The preceding theory can be extended further for more complicated transient inputs whenever the input can be described by the sum of several step functions.

If a particular input, for instance, can be described as

$$\ddot{\theta}_i(t) = \sum_n A_n \, u(t - T_n) \quad , \tag{65}$$

which is a transient acceleration input composed of n separate steps of different magnitudes A_n, and initiated at times T_n, respectively, (the function $u(t - T_n)$ represents a unit step function initiated at time T_n) the canal response in this case would be the following.

$$\dot{\theta}_m(t) = \sum_n A_n \, f_3(t - T_n) \quad . \tag{66}$$

5. Canal Response to Suddenly Applied Sinusoidal Velocity Input

The canal response to a much larger class of input functions can be evaluated with the additional knowledge of its response to suddenly applied sinusoidal inputs.

In the first case of such inputs, the input function considered is a sinusoidal velocity input, or

$$\dot{\theta}_i(t) = \Omega \sin \omega t \tag{67}$$

with (see Appendix),

$$\dot{\theta}_i(s) = \Omega \frac{\omega}{s^2 + \omega^2} \quad . \tag{68}$$

Using Equation (31), the canal response is obtained as

$$\theta_e(s) = \frac{s}{(s + \omega_1)(s + \omega_2)} \Omega \frac{\omega}{s^2 + \omega^2} \tag{69}$$

which may be re-written as

$$\theta_e(s) = \Omega \frac{s\omega}{(s + \omega_1)(s + \omega_2)(s - j\omega)(s + j\omega)} \tag{70}$$

which, expanded into partial fractions, becomes

$$\theta_e(s) = \Omega \left[- \frac{\omega \omega_1}{(\omega_2 - \omega_1)(\omega_1^2 + \omega^2)} \frac{1}{s + \omega_1} + \frac{\omega \omega_2}{(\omega_2 - \omega_1)(\omega_2^2 + \omega^2)} \frac{1}{s + \omega_2} \right.$$
$$+ j \frac{\omega^2}{(\omega_1 + j\omega)(\omega_2 + j\omega)(2j\omega)} \frac{1}{s - j\omega}$$
$$\left. - j \frac{\omega^2}{(\omega_1 - j\omega)(\omega_2 - j\omega)(-2j\omega)} \frac{1}{s + j\omega} \right] . \tag{71}$$

Taking the inverse Laplace transform,

$$\theta_e(t) = \Omega \left[- \frac{\omega \omega_1}{(\omega_2 - \omega_1)(\omega_1^2 + \omega^2)} e^{-\omega_1 t} + \frac{\omega \omega_2}{(\omega_2 - \omega_1)(\omega_2^2 + \omega^2)} e^{-\omega_2 t} \right.$$
$$\left. + \frac{\omega}{2} \left\{ \frac{1}{(\omega_1 + j\omega)(\omega_2 + j\omega)} e^{j\omega t} + \frac{1}{(\omega_1 - j\omega)(\omega_2 - j\omega)} e^{-j\omega t} \right\} \right] \tag{72}$$

Equation (72) may be re-written as

$$\theta_e(t) = \Omega \left[- \frac{\omega\,\omega_1}{(\omega_2 - \omega_1)(\omega_1^2 + \omega^2)} e^{-\omega_1 t} + \frac{\omega\,\omega_2}{(\omega_2 - \omega_1)(\omega_2^2 + \omega^2)} e^{-\omega_2 t} \right.$$

$$+ \frac{\omega}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \left\{ \frac{\omega_1\omega_2 - \omega^2}{2} (e^{j\omega t} + e^{-j\omega t}) + \frac{\omega(\omega_1 + \omega_2)}{2j} \right.$$

$$\left. \left. (e^{j\omega t} - e^{-j\omega t}) \right\} \right] \tag{73}$$

or,

$$\theta_e(t) = \Omega \frac{\omega}{\omega_2 - \omega_1} \left[- \frac{\omega_1}{\omega_1^2 + \omega^2} e^{-\omega_1 t} + \frac{\omega_2}{\omega_2^2 + \omega^2} e^{-\omega_2 t} \right.$$

$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t + \frac{\omega(\omega_2^2 - \omega_1^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t \right]. \tag{74}$$

From the prior definition of <u>measured velocity</u>,

$$\dot{\theta}_m(t) = L\,\theta_e(t) = \Omega \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1} \left[- \frac{\omega_1}{\omega_1^2 + \omega^2} e^{-\omega_1 t} + \frac{\omega_2}{\omega_2^2 + \omega^2} e^{-\omega_2 t} \right.$$

$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t + \frac{\omega(\omega_2^2 - \omega_1^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t \right]. \tag{75}$$

It may be of interest to investigate the steady-state form of the response $\dot{\theta}_m(t)$ (when the exponential terms $e^{-\omega_1 t}$ and $e^{-\omega_2 t}$ have decayed to insignificantly low levels).

Thus,

$$\dot{\theta}_m(t)_{ss} = \Omega \left[\frac{\omega(\omega_2 + \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t + \frac{\omega^2(\omega_2 + \omega_1)^2}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t \right] \quad (76)$$

or,

$$\dot{\theta}_m(t)_{ss} = \Omega \frac{\omega(\omega_2 + \omega_1)}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} \left[\frac{\omega_1\omega_2 - \omega^2}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} \cos \omega t + \frac{\omega(\omega_2 + \omega_1)}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} \sin \omega t \right]. \quad (77)$$

By letting

$$\frac{\omega_1\omega_2 - \omega^2}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} = \cos \phi \quad (78)$$

and

$$\frac{\omega(\omega_2 + \omega_1)}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} = \sin \phi \quad , \quad (79)$$

Equation (77) becomes

$$\dot{\theta}_m(t)_{ss} = \Omega (\omega_2 + \omega_1) \frac{\omega}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} \cos (\omega t - \phi) \quad (80)$$

or,

$$\dot{\theta}_m(t)_{ss} = \Omega (\omega_2 + \omega_1) \frac{\omega}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} \sin (\omega t + \frac{\pi}{2} - \phi) . \quad (81)$$

From Equations (21) and (23), Equation (81) can be re-written as

$$\dot{\theta}_m(t)_{ss} = \Omega L A_{\dot{\theta}_i}(\omega) \sin (\omega t + \frac{\pi}{2} - \phi) . \quad (82)$$

From Equation (26),

$$B_{\dot{\theta}_i}(\omega) = \frac{\pi}{2} - \phi_1 - \phi_2 \quad (83)$$

where

$$\phi_1 = \tan^{-1} \frac{\omega}{\omega_1} \tag{84}$$

and

$$\phi_2 = \tan^{-1} \frac{\omega}{\omega_2} \quad , \tag{85}$$

from which

$$\tan \phi_1 = \frac{\omega}{\omega_1} \tag{86}$$

$$\tan \phi_2 = \frac{\omega}{\omega_2} \tag{87}$$

and $$\tan (\phi_1 + \phi_2) = \frac{\frac{\omega}{\omega_1} + \frac{\omega}{\omega_2}}{1 - \frac{\omega}{\omega_1} \frac{\omega}{\omega_2}} = \frac{\omega(\omega_2 + \omega_1)}{\omega_1\omega_2 - \omega^2} \tag{88}$$

and $$\cos (\phi_1 + \phi_2) = \frac{1}{\left[\frac{\omega^2 (\omega_2 + \omega_1)^2}{(\omega_1\omega_2 - \omega^2)^2} + 1\right]^{\frac{1}{2}}} = \frac{\omega_1\omega_2 - \omega^2}{\left[(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)\right]^{\frac{1}{2}}} \; . \tag{89}$$

Thus, referring back to Equation (78), it is seen that

$$\phi_1 + \phi_2 = \phi \tag{90}$$

and, that from Equation (83),

$$\frac{\pi}{2} - \phi = B_{\dot{\theta}_i} (\omega) \; . \tag{91}$$

Finally, from Equation (82)

$$\dot{\theta}_m(t)_{ss} = L \times A_{\dot{\theta}_i} (\omega) \; \Omega \sin \left(\omega t + B_{\dot{\theta}_i} (\omega)\right). \tag{92}$$

This is exactly what would be obtained by using the frequency response functions previously derived, and the definition of measured velocity.

In particular, if the frequency of the input function is chosen such that

$$\omega^2 = \omega_1 \omega_2 \quad , \tag{93}$$

which corresponds to the natural frequency of the canal, then, from Equation (78),

$$\cos \phi = 0 \tag{94}$$

or

$$\phi = \frac{\pi}{2} \quad . \tag{95}$$

In this case Equation (81) becomes

$$\dot{\theta}_m(t)_{ss} = \Omega \frac{\sqrt{\omega_1\omega_2}\,(\omega_2 + \omega_1)}{\left[(\omega_1^2 + \omega_1\omega_2)(\omega_2^2 + \omega_1\omega_2)\right]^{\frac{1}{2}}} \sin\left(\omega t + \frac{\pi}{2} - \frac{\pi}{2}\right) \tag{96}$$

or

$$\dot{\theta}_m(t)_{ss} = \Omega \frac{\left[\omega_1\omega_2\,(\omega_2 + \omega_1)^2\right]^{\frac{1}{2}}}{\left[\omega_1\omega_2\,(\omega_2 + \omega_1)^2\right]^{\frac{1}{2}}} \sin \omega t \quad , \tag{97}$$

or

$$\dot{\theta}_m(t)_{ss} = \Omega \sin \omega t = \dot{\theta}_i(t) \quad . \tag{98}$$

Thus, at this particular frequency, the steady-state response function $\dot{\theta}_m(t)_{ss}$ is, indeed, an exact measurement of the input velocity $\dot{\theta}_i(t)$.

Returning to transient response considerations, the integral and differential functions $\theta_m(t)$ and $\ddot{\theta}_m(t)$ for the input under study may be obtained from Equation (75) with the assumption that $\theta_m(o) = 0$.

Thus,

$$\ddot{\theta}_m(t) = \Omega \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1} \left[\frac{\omega_1^{\,2}}{\omega_1^{\,2} + \omega^2} e^{-\omega_1 t} - \frac{\omega_2^{\,2}}{\omega_2^{\,2} + \omega^2} e^{-\omega_2 t} - \frac{\omega(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^{\,2} + \omega^2)(\omega_2^{\,2} + \omega^2)} \sin \omega t + \frac{\omega^2(\omega_2^{\,2} - \omega_1^{\,2})}{(\omega_1^{\,2} + \omega^2)(\omega_2^{\,2} + \omega^2)} \cos \omega t \right] \tag{99}$$

and

$$\theta_m(t) = \Omega \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1} \left[\frac{1}{\omega_1^{\,2} + \omega^2} e^{-\omega_1 t} - \frac{1}{\omega_2^{\,2} + \omega^2} e^{-\omega_2 t} + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{\omega(\omega_1^{\,2} + \omega^2)(\omega_2^{\,2} + \omega^2)} \sin \omega t - \frac{(\omega_2^{\,2} - \omega_1^{\,2})}{(\omega_1^{\,2} + \omega^2)(\omega_2^{\,2} + \omega^2)} \cos \omega t \right]. \tag{100}$$

6. Canal Response to Suddenly Applied Sinusoidal Acceleration Input

The input function in this case is

$$\ddot{\theta}_i(t) = A \sin \omega t \tag{101}$$

from which

$$\ddot{\theta}_i(s) = A \frac{\omega}{s^2 + \omega^2} . \tag{102}$$

Using Equation (39), the following response function is obtained.

$$\theta_e(s) = \frac{1}{(s + \omega_1)(s + \omega_2)} \quad A \frac{\omega}{s^2 + \omega^2} \tag{103}$$

which may be re-written as

$$\theta_e(s) = A \frac{\omega}{(s + \omega_1)(s + \omega_2)(s - j\omega)(s + j\omega)} . \tag{104}$$

Expanding by partial fraction,

$$\theta_e(s) = A\left[\frac{\omega}{(\omega_2 - \omega_1)(\omega_1^2 + \omega^2)}\ \frac{1}{s+\omega_1} - \frac{\omega}{(\omega_2 - \omega_1)(\omega_2^2 + \omega^2)}\ \frac{1}{s+\omega_2}\right.$$
$$\left. + \frac{\omega}{(\omega_1 + j\omega)(\omega_2 + j\omega)(2j\omega)}\ \frac{1}{s - j\omega} + \frac{\omega}{(\omega_1 - j\omega)(\omega_2 - j\omega)(-2j\omega)}\ \frac{1}{s + j\omega}\right]. \tag{105}$$

Taking the inverse Laplace transformation,

$$\theta_e(t) = A\left[\frac{\omega}{(\omega_2 - \omega_1)(\omega_1^2 + \omega^2)}\ e^{-\omega_1 t} - \frac{\omega}{(\omega_2 - \omega_1)(\omega_2^2 + \omega^2)}\ e^{-\omega_2 t}\right.$$
$$\left. + \frac{1}{2j}\left\{\frac{1}{(\omega_1 + j\omega)(\omega_2 + j\omega)}\ e^{j\omega t} - \frac{1}{(\omega_1 - j\omega)(\omega_2 - j\omega)}\ e^{-j\omega t}\right\}\right]. \tag{106}$$

This may be re-written as

$$\theta_e(t) = A\left[\frac{\omega}{(\omega_2 - \omega_1)(\omega_1^2 + \omega^2)}\ e^{-\omega_1 t} - \frac{\omega}{(\omega_2 - \omega_1)(\omega_2^2 + \omega^2)}\ e^{-\omega_2 t}\right.$$
$$\left. + \frac{1}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\left\{(\omega_1\omega_2 - \omega^2)\ \frac{e^{j\omega t} - e^{-j\omega t}}{2j} - \omega(\omega_2 + \omega_1)\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right\}\right] \tag{107}$$

or,

$$\theta_e(t) = A\ \frac{\omega}{\omega_2 - \omega_1}\left[\frac{1}{\omega_1^2 + \omega^2} e^{-\omega_1 t} - \frac{1}{\omega_2^2 + \omega^2}\ e^{-\omega_2 t}\right.$$
$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{\omega(\omega_1^2 - \omega^2)(\omega_2^2 + \omega^2)}\ \sin\omega t - \frac{\omega_2^2 - \omega_1^2}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\ \cos\omega t\right]. \tag{108}$$

The measured velocity function in this case will be

$$\dot{\theta}_m(t) = L\,\theta_e(t) = A\,\frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[\frac{1}{\omega_1^2 + \omega^2}\,e^{-\omega_1 t} - \frac{1}{\omega_2^2 + \omega^2}\,e^{-\omega_2 t}\right.$$

$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{\omega(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\sin\omega t - \frac{\omega_2^2 - \omega_1^2}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\cos\omega t\right]. \quad (109)$$

By differentiating and integrating Equation (109) (again with the assumption of $\theta_m(o) = 0$),

$$\ddot{\theta}_m(t) = A\,\frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[-\frac{\omega_1}{\omega_1^2 + \omega^2}\,e^{-\omega_1 t} + \frac{\omega_2}{\omega_2^2 + \omega^2}\,e^{-\omega_2 t}\right.$$

$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\cos\omega t + \frac{\omega(\omega_2^2 - \omega_1^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\sin\omega t\right] \quad (110)$$

and

$$\theta_m(t) = A\,\frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[-\frac{1}{\omega_1(\omega_1^2 + \omega^2)}\,e^{-\omega_1 t} + \frac{1}{\omega_2(\omega_2^2 + \omega^2)}\,e^{-\omega_2 t}\right.$$

$$- \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{\omega^2(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\cos\omega t - \frac{\omega_2^2 - \omega_1^2}{\omega(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)}\sin\omega t$$

$$\left. + \frac{\omega_2 - \omega_1}{\omega_1\omega_2\omega^2}\right]. \quad (111)$$

7. Summary of Response to Suddenly Applied Sinusoidal Inputs of Velocity and Acceleration

As was the case for the response of the semicircular canal to step inputs, only four basic functions have been derived for sinusoidal transient inputs; namely,

$$f_5(t) = \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[\frac{\omega_1^2}{\omega_1^2 + \omega^2} e^{-\omega_1 t} - \frac{\omega_2^2}{\omega_2^2 + \omega^2} e^{-\omega_2 t}\right.$$

$$\left. - \frac{\omega(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t + \frac{\omega^2(\omega_2^2 - \omega_1^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t\right] \quad (112)$$

$$f_6(t) = \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[-\frac{\omega_1}{\omega_1^2 + \omega^2} e^{-\omega_1 t} + \frac{\omega_2}{\omega_2^2 + \omega^2} e^{-\omega_2 t}\right.$$

$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t + \frac{\omega(\omega_2^2 - \omega_1^2)}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t\right] \quad (113)$$

$$f_7(t) = \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[\frac{1}{\omega_1^2 + \omega^2} e^{-\omega_1 t} - \frac{1}{\omega_2^2 + \omega^2} e^{-\omega_2 t}\right.$$

$$\left. + \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{\omega(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t - \frac{\omega_2^2 - \omega_1^2}{(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t\right] \quad (114)$$

$$f_8(t) = \frac{\omega(\omega_2 + \omega_1)}{\omega_2 - \omega_1}\left[-\frac{1}{\omega_1(\omega_1^2 + \omega^2)} e^{-\omega_1 t} + \frac{1}{\omega_2(\omega_2^2 + \omega^2)} e^{-\omega_2 t} + \frac{\omega_2 - \omega_1}{\omega_1\omega_2\omega^2}\right.$$

$$\left. - \frac{(\omega_2 - \omega_1)(\omega_1\omega_2 - \omega^2)}{\omega^2(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \cos \omega t - \frac{\omega_2^2 - \omega_1^2}{\omega(\omega_1^2 + \omega^2)(\omega_2^2 + \omega^2)} \sin \omega t\right]. \quad (115)$$

For a suddenly applied sinusoidal velocity input of amplitude Ω,

$$\ddot{\theta}_m(t) = \Omega\ f_5(t) \quad (116)$$

$$\dot{\theta}_m(t) = \Omega\ f_6(t) \quad (117)$$

$$\theta_m(t) = \Omega\ f_7(t)\ . \quad (118)$$

For a suddenly applied sinusoidal acceleration input of amplitude A,

$$\ddot{\theta}_m(t) = A\, f_6(t) \tag{119}$$

$$\dot{\theta}_m(t) = A\, f_7(t) \tag{120}$$

$$\theta_m(t) = A\, f_8(t) \quad . \tag{121}$$

The four functions $f_5(t)$, $f_6(t)$, $f_7(t)$, and $f_8(t)$ are presented in Figures 8, 9, 10, and 11, respectively. Only one set of conditions is presented in each figure. This set of conditions is as follows.

$\omega_1 = 0.1$ $\qquad$ $L = 10.1$

$\omega_2 = 10.0$ $\qquad$ $P = 1.0$

$\omega = \frac{\pi}{2}$ radians per second, corresponding to a period of four seconds for the sinusoidal input function.

8. Canal Response to Sinusoidal Inputs of Finite Duration

To illustrate how the functions $f_5(t)$, $f_6(t)$, $f_7(t)$, and $f_8(t)$ may be used to obtain the canal response to more complicated forms of input transients, consider the following possible experiment.

Assume a subject is to be subjected to a test in which he is to be accelerated through only one complete cycle of sinusoidal acceleration. The input function can then be described as,

$$\ddot{\theta}_i(t) = A \left[u(t) \sin \omega t - u(t - T) \sin \omega t \right] \tag{122}$$

where T is the period of oscillation, or $T = \frac{2\pi}{\omega}$.

This describes an acceleration input of sinusoidal form impressed at time t = 0 and continuing thereafter, plus a second sinusoidal input of the same amplitude, and opposite phase impressed at the completion

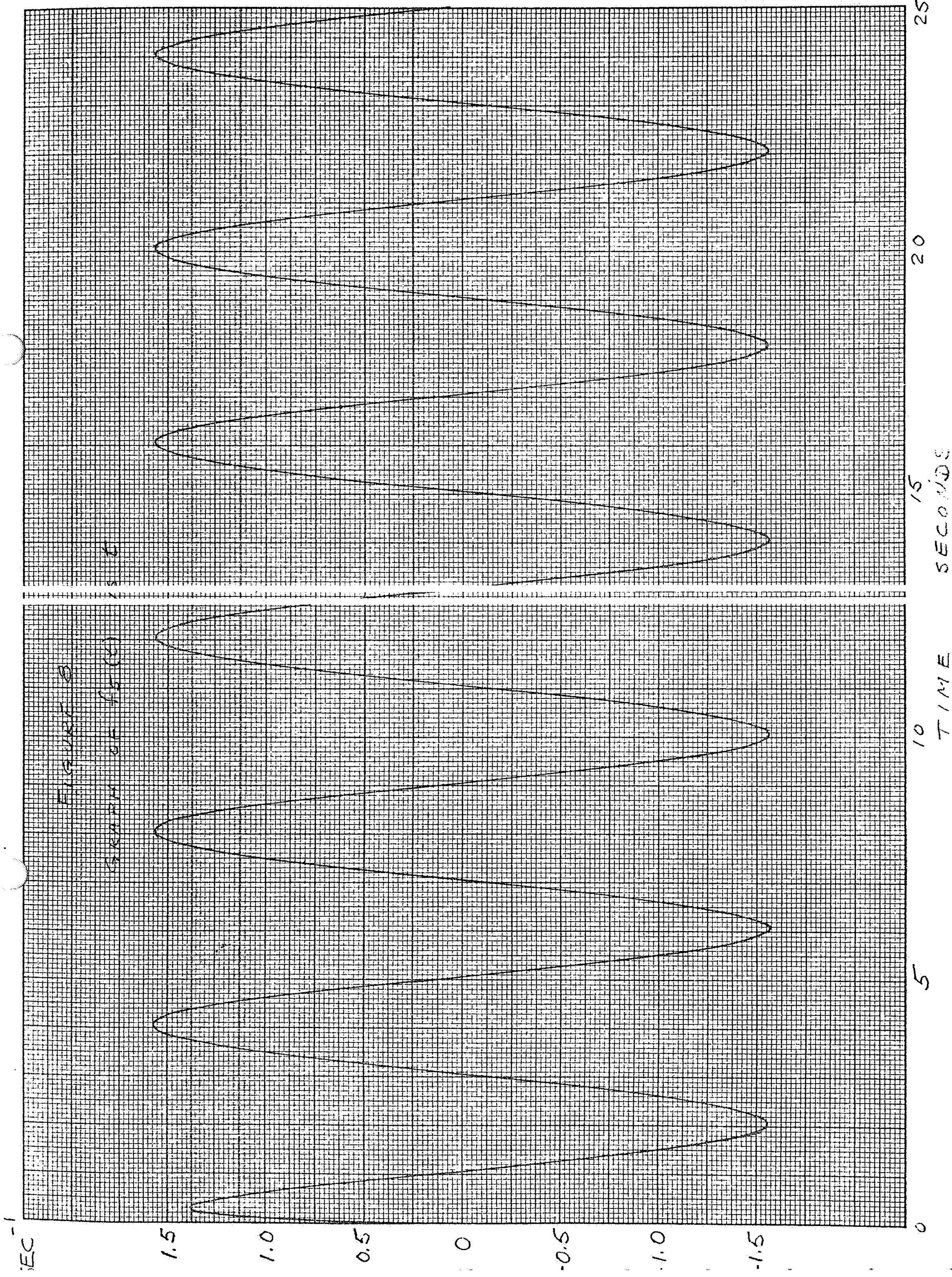
FIGURE 8
GRAPH OF f5(t) VS t
SEC⁻¹
1.5
1.0
0.5
0
-0.5
-1.0
-1.5
0
5
10
15
20
25
TIME SECONDS

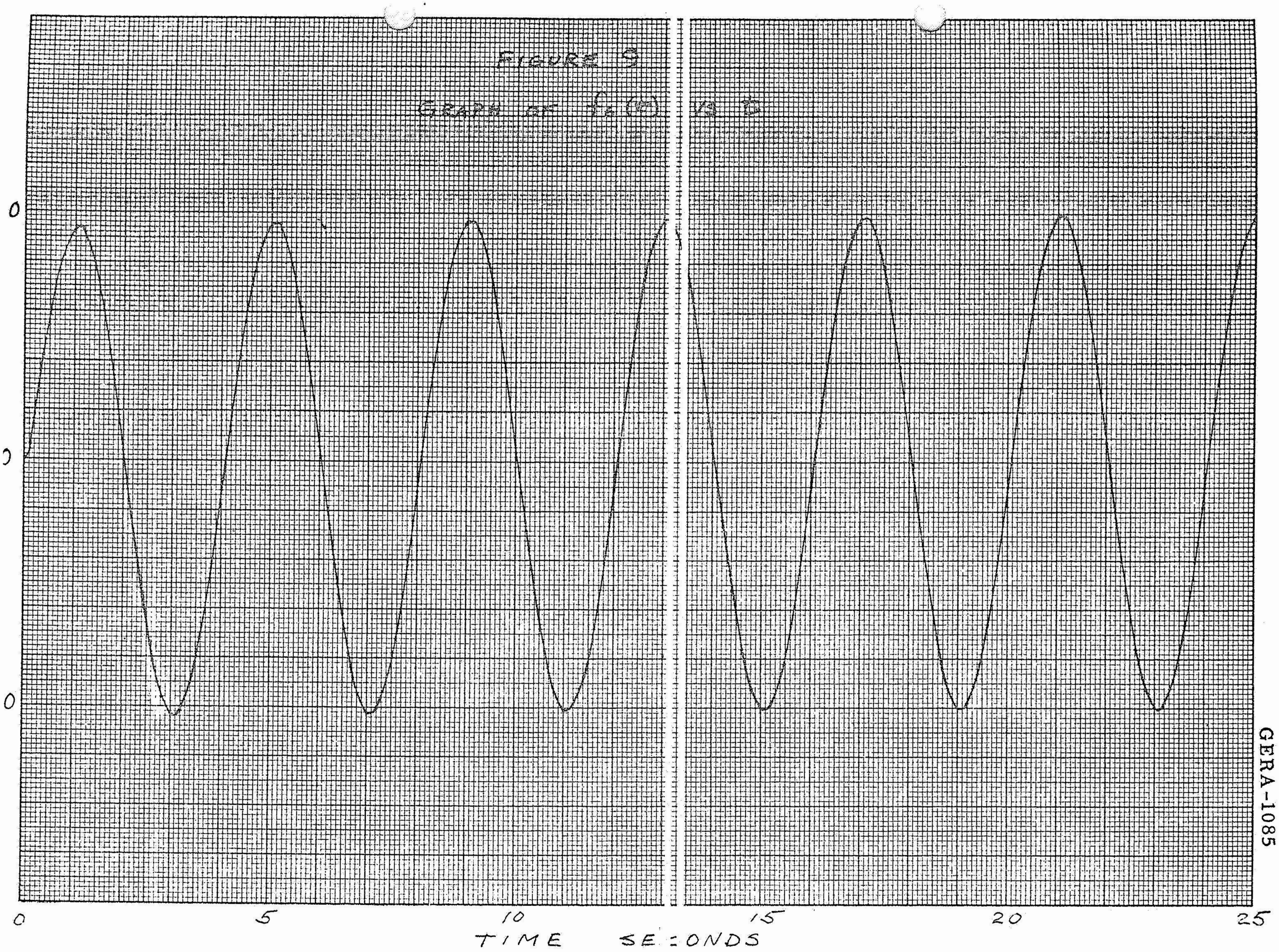
FIGURE 9
GRAPH OF $f_s(t)$ VS t
0
0
0
5
10
15
20
25
TIME SECONDS

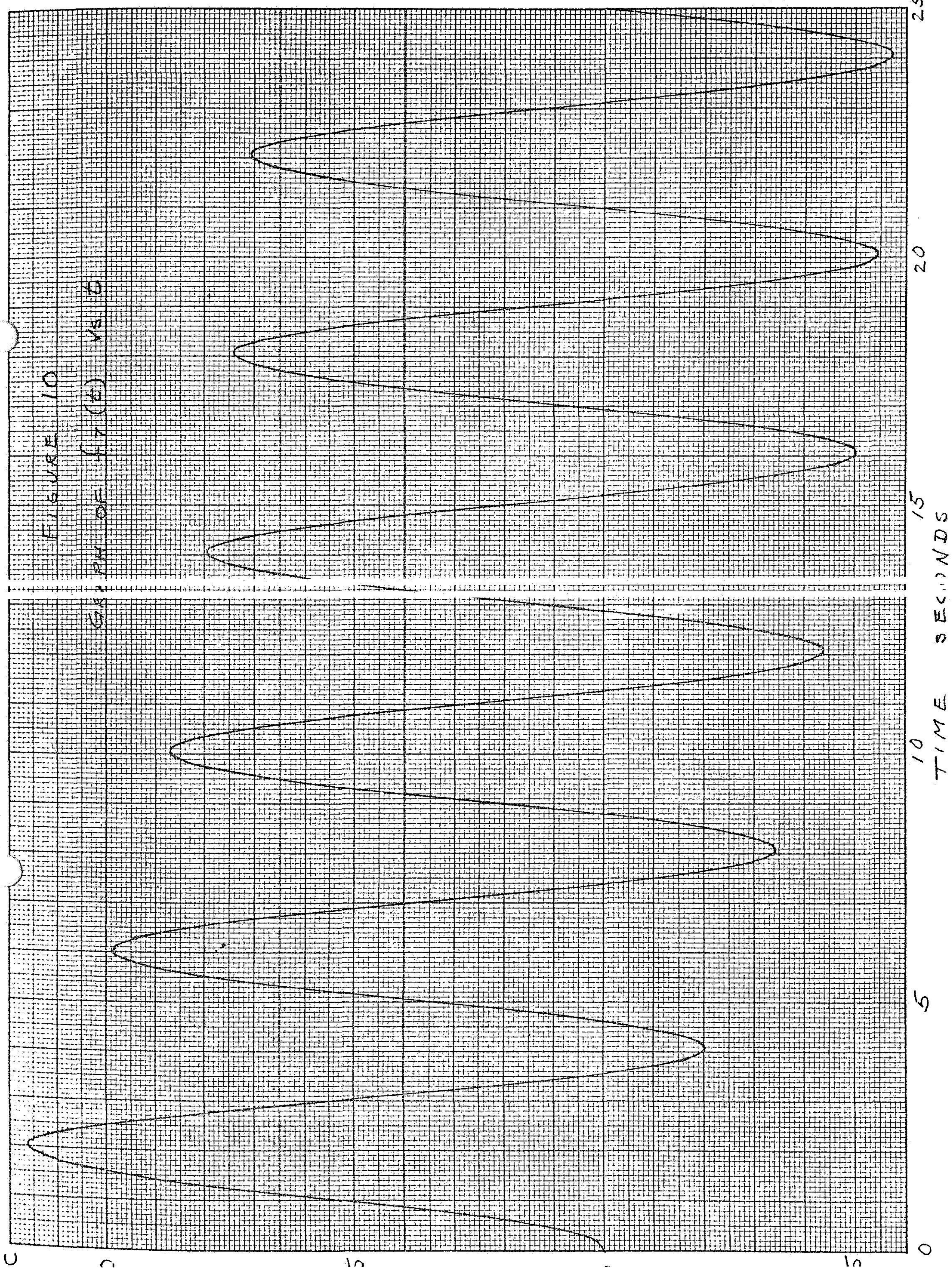
FIGURE 10
GRAPH OF f7(t) vs t
TIME SECONDS
0
5
10
15
20
25

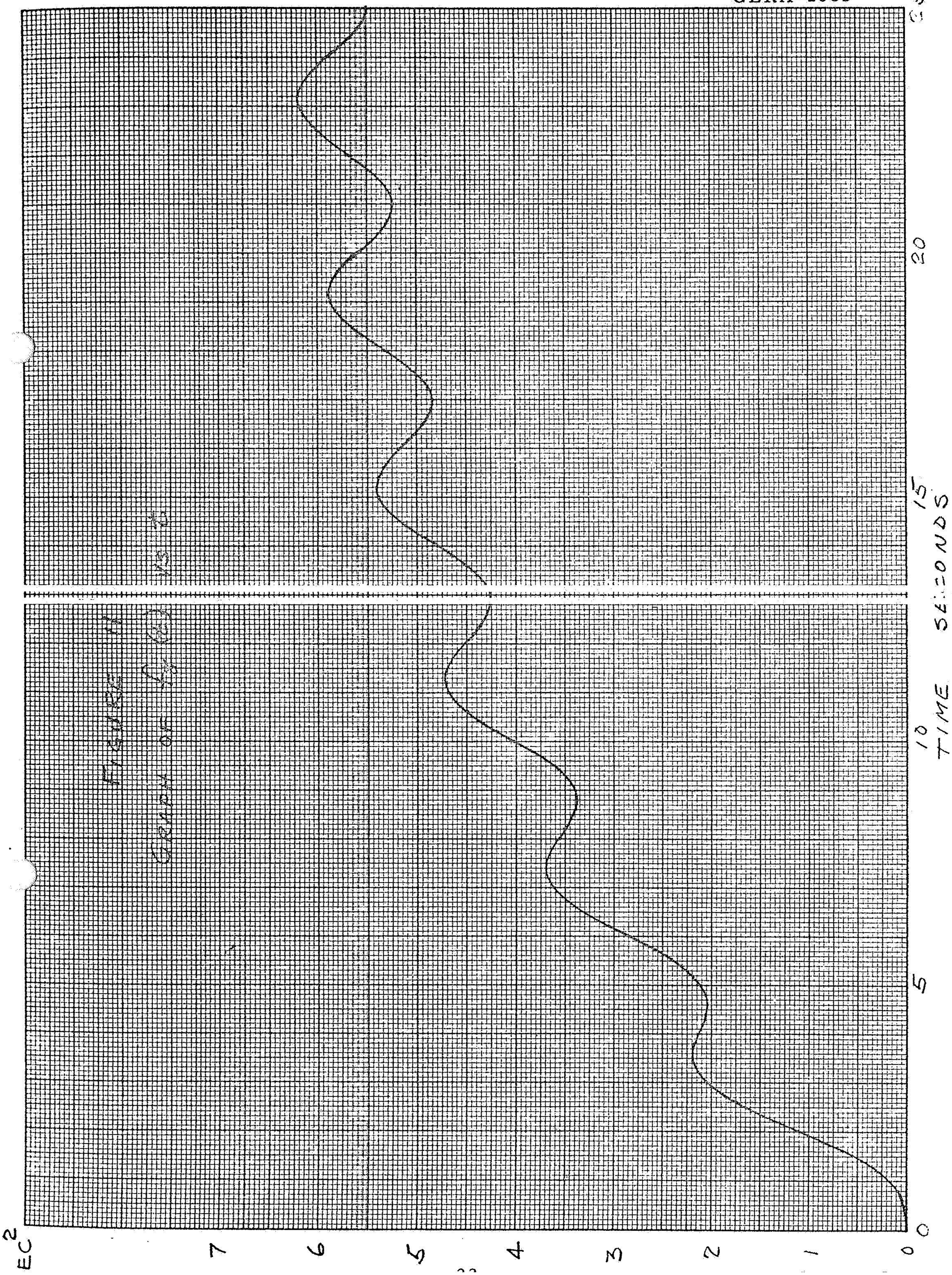
FIGURE 11
GRAPH OF $L_y(t)$ vs t
EC^2
0
1
2
3
4
5
6
7
0
5
10
15
20
25
TIME SECONDS

of the first cycle of the first input. Thus, these two sinusoidal functions cancel each other exactly for all times following $t = T$.

The response functions to this type input can be written as,

$$\ddot{\theta}_m(t) = A\left[f_6(t) - f_6(t - T)\right] \tag{123}$$

$$\dot{\theta}_m(t) = A\left[f_7(t) - f_7(t - T)\right] \tag{124}$$

$$\theta_m(t) = A\left[f_8(t) - f_8(t - T)\right]. \tag{125}$$

Since, as indicated previously, $f(t) = 0$ for $t < o$, or $f(t - T) = 0$ for $t < T$, the response functions for $o < t < T$ will be identical to those described by Equations (119), (120), and (121). For $T \leq t$, the response functions will be as described by Equations (123), (124), and (125).

The input functions for $o < t < T$ will be,

$$\ddot{\theta}_i(t) = A \sin \omega t \tag{126}$$

$$\dot{\theta}_i(t) = A \frac{1}{\omega}(1 - \cos \omega t) \tag{127}$$

$$\theta(t) = A \frac{1}{\omega}(t - \frac{1}{\omega} \sin \omega t), \tag{128}$$

and for $T \leq t$,

$$\ddot{\theta}_i(t) = 0 \tag{129}$$

$$\dot{\theta}_i(t) = 0 \tag{130}$$

$$\theta_i(t) = A \frac{T}{\omega} \tag{131}$$

or, since $T = \frac{2\pi}{\omega}$,

$$\theta_i(t) = A \frac{2\pi}{\omega^2}. \tag{132}$$

In Figure 12 are shown the dimension less input and output functions $\frac{\ddot{\theta}_i(t)}{A}$ and $\frac{\ddot{\theta}_m(t)}{A}$. The assumed values for the canal parameters are again $\omega_1 = 0.1$, $\omega_2 = 10.0$, and the frequency $\omega = \frac{\pi}{2}$ radians/second.

For the same parameters, Figure 13 presents the input and output functions $\dot{\theta}_i(t)/A$ and $\dot{\theta}_m(t)/A$.

Input and output functions $\theta_i(t)/A$ and $\theta_m(t)/A$ are shown in Figure 14.

DISCUSSION

As pointed out earlier in the section covering the frequency response of the semicircular canals, they may be considered as accelerometers, velocity meters, or position meters, depending on the frequency region of interest. In particular, the range of frequency from ω_1 to ω_2, where the semicircular canals behave as velocity meters, seems to cover quite well the range of frequency involved in normal body movements.

These canal characteristics can also be demonstrated by a transient response analysis.

Assuming a step input of acceleration

$$\ddot{\theta}_i(t) = A \tag{133}$$

with

$$\dot{\theta}_i(t) = At \tag{134}$$

and

$$\theta_i(t) = A\,\tfrac{1}{2}t^2 , \tag{135}$$

the canal response in terms of endolymph displacement $\theta_e(t)$ relative to the canal is, as shown by Equation (44),

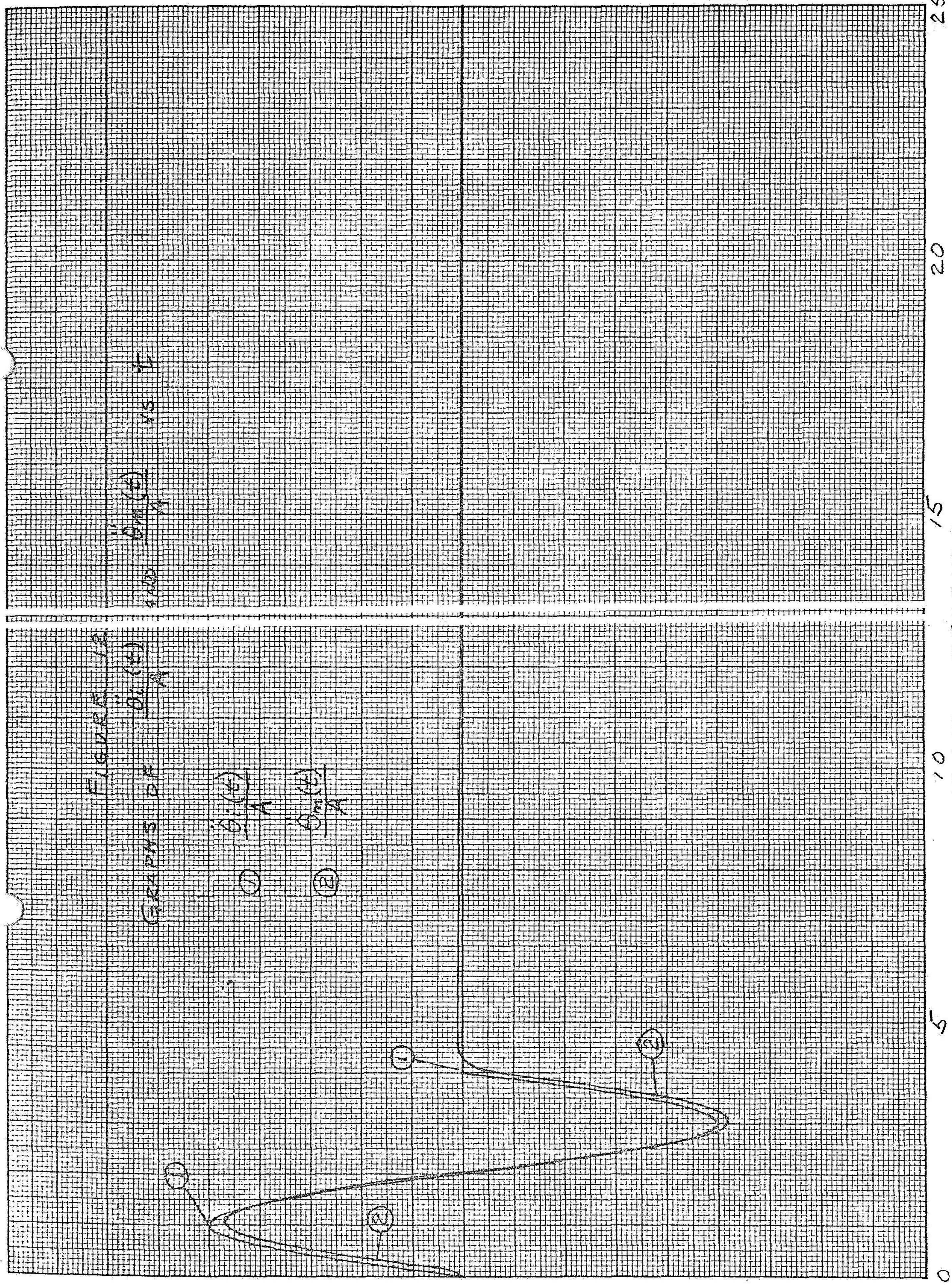

FIGURE 12
GRAPHS OF $\frac{\ddot{\theta}_i(t)}{A}$ AND $\frac{\ddot{\theta}_m(t)}{A}$ VS t
① $\frac{\dot{\theta}_i(t)}{A}$
② $\frac{\dot{\theta}_m(t)}{A}$
TIME SECONDS
0
5
10
15
20
25
1
0
-1

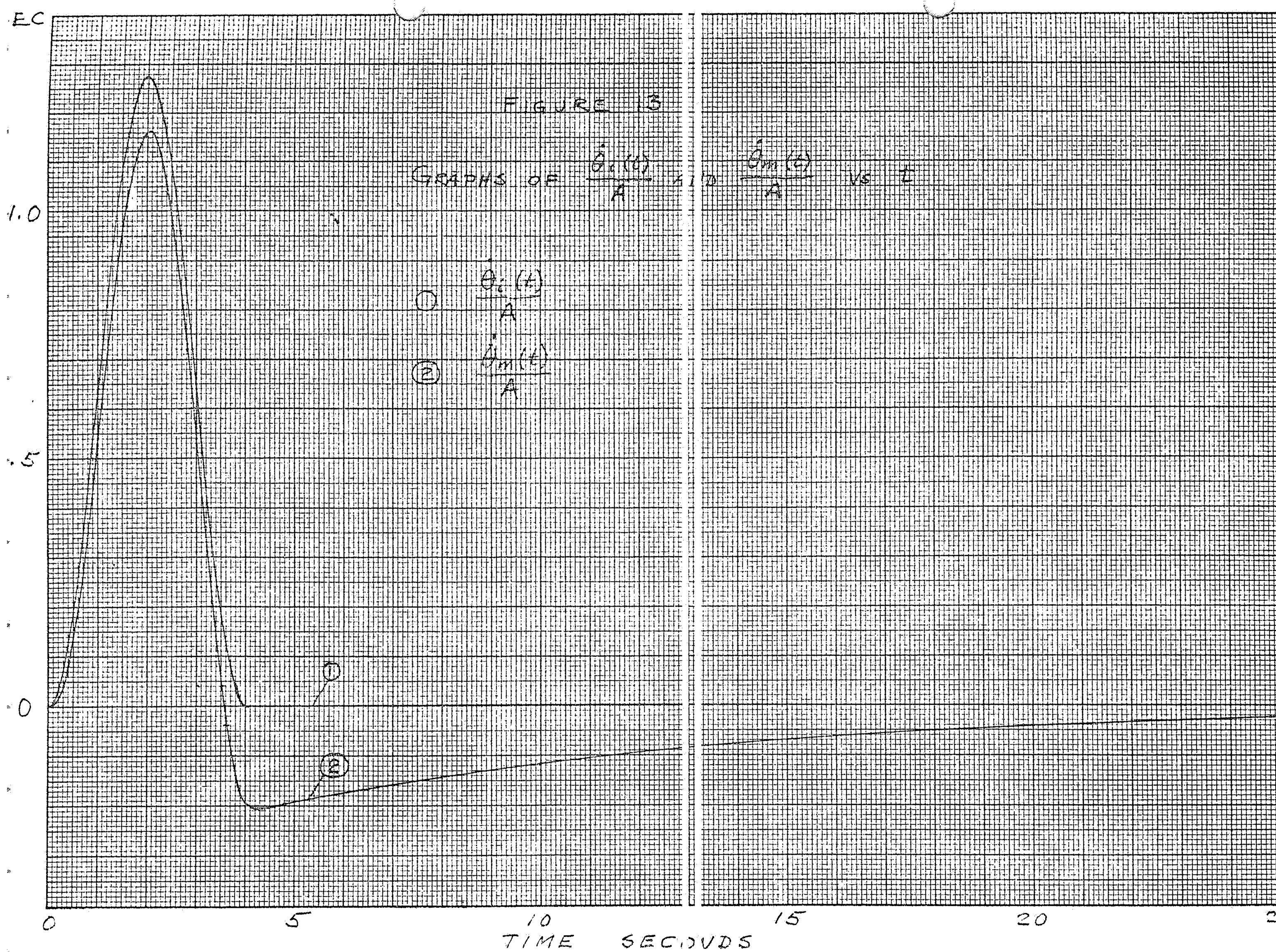
FIGURE 13
GRAPHS OF $\frac{\dot{\theta}_i(t)}{A}$ AND $\frac{\dot{\theta}_m(t)}{A}$ VS t
1 $\frac{\dot{\theta}_i(t)}{A}$
2 $\frac{\dot{\theta}_m(t)}{A}$
1
2
EC
1.0
.5
0
0
5
10
15
20
25
TIME SECONDS

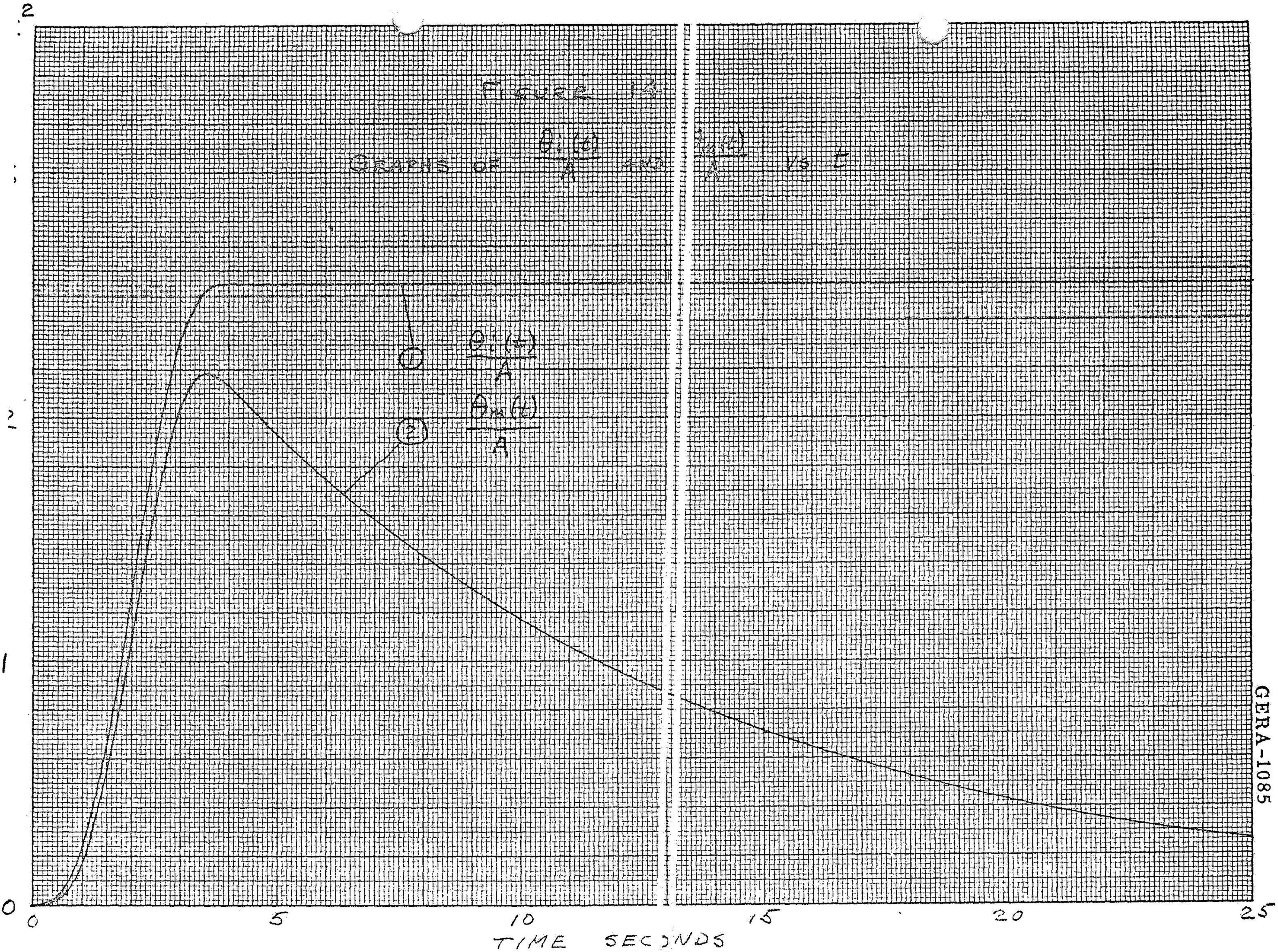
FIGURE 14
GRAPHS OF $\frac{\theta_i(t)}{A}$ AND $\frac{\theta_m(t)}{A}$ VS t
1 $\frac{\theta_i(t)}{A}$
2 $\frac{\theta_m(t)}{A}$
0
1
2
0
5
10
15
20
25
TIME SECONDS

$$\theta_e(t) = A \frac{1}{\omega_2 - \omega_1} \left(- \frac{1}{\omega_1} e^{-\omega_1 t} + \frac{1}{\omega_2} e^{-\omega_2 t} + \frac{\omega_2 - \omega_1}{\omega_2 \omega_1} \right) . \tag{136}$$

Now, if the canal were an accelerometer with scale factor $\frac{1}{P}$, its response would be

$$\frac{\ddot{\theta}_i(t)}{P} = \frac{A}{P} . \tag{137}$$

If it were a velocity meter with a scale factor $\frac{1}{L}$, its response would be

$$\frac{\dot{\theta}_i(t)}{L} = \frac{At}{L} . \tag{138}$$

If it were a position meter with a scale factor of unity, its response would be

$$\theta_i(t) = A \frac{t^2}{2} . \tag{139}$$

These assumed responses are shown in Figure 15. This figure uses a log-log presentation in order to cover with sufficient detail a large range of time and a large range of ordinate. The ordinate scale numbers are in units of second squared, and must be multiplied by A to represent the angular displacement response. Also shown in this figure is the actual response function, $\theta_e(t)$.

The assumed canal constants are: $\omega_1 = 0.1$, $\omega_2 = 200$, $L = 200.1$, and $P = 20$.

It can be seen in this figure that the actual response $\theta_e(t)$ is very nearly equal to $\theta_i(t)$ for times from 0 to 0.01 second. In the time interval between 0.01 second and 10 seconds, the actual response $\theta_e(t)$ is very nearly equal to $\frac{\dot{\theta}_i(t)}{L}$ and when time exceeds 10 seconds, $\theta_e(t)$ is very nearly equal to $\frac{\ddot{\theta}_i(t)}{P}$.

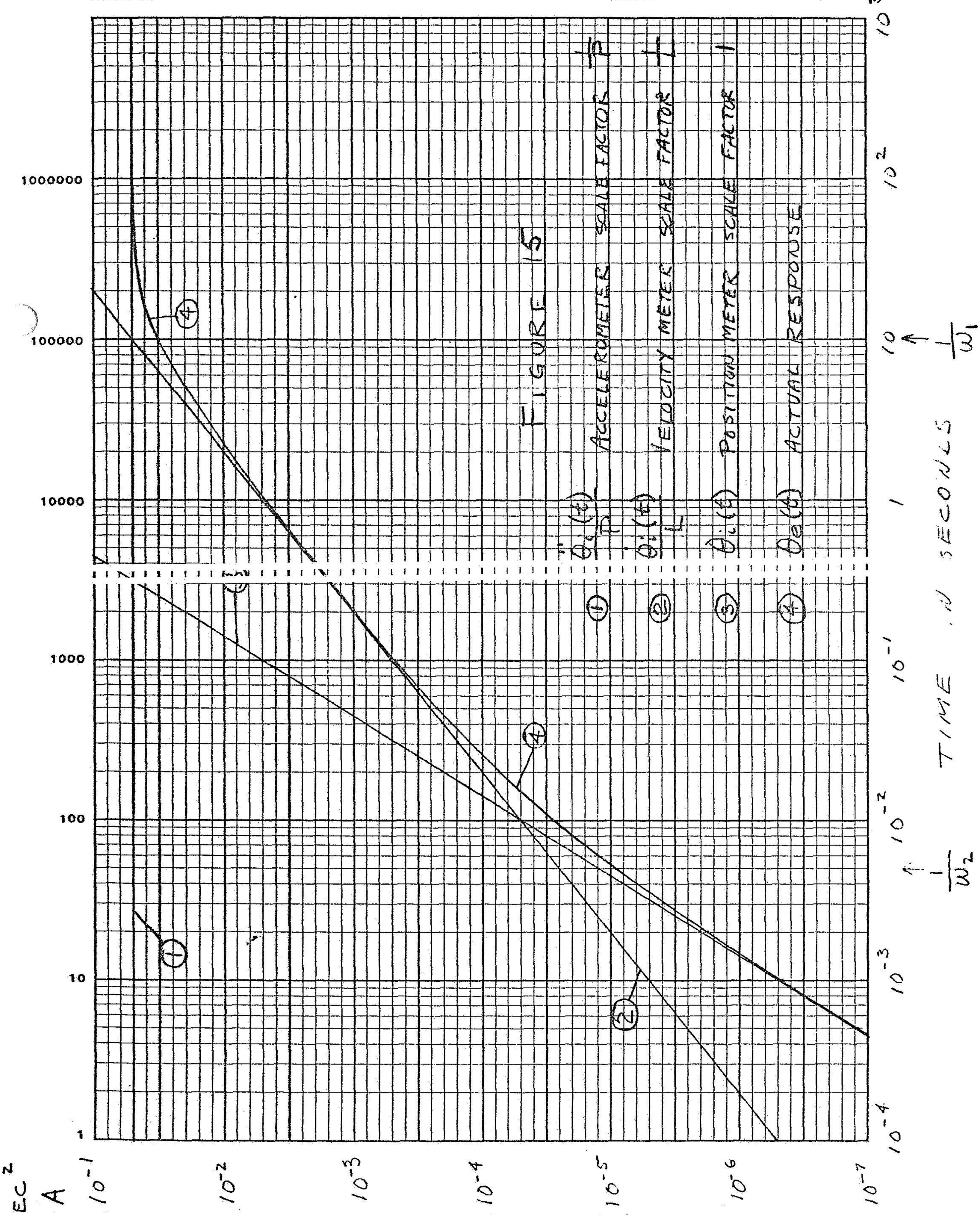
FIGURE 15
① θi(t)/P ACCELEROMETER SCALE FACTOR P
② θi(t)/L VELOCITY METER SCALE FACTOR L
③ θi(t) POSITION METER SCALE FACTOR 1
④ θe(t) ACTUAL RESPONSE
TIME IN SECONDS
1/ω1
1/ω2
SEC² x A
1000000
100000
10000
1000
100
10
1
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
10^3
10^2
10
1
10^-1
10^-2
10^-3
10^-4

Thus, for the first few milliseconds, the semicircular canal responds to the step of input acceleration as a position meter. Then, during the time interval from approximately $\frac{1}{\omega_2}$ to $\frac{1}{\omega_1}$ seconds, the semicircular canal response becomes that of a velocity meter. For longer times (in excess of $\frac{1}{\omega_1}$ seconds) the semicircular canal response approximates that of an accelerometer.

Therefore, for normal body movements, a single integration process on the semicircular canal signal is sufficient to provide a reasonably accurate knowledge of position. However, if a long term (say, for 20 seconds or longer) memory of position were required from semicircular canal data (assuming visual or auditory cues were unavailable), a second integration process would be necessary since the semicircular canal is behaving as an accelerometer for those longer intervals of time.

This is well demonstrated by Figure 14, where, on the basis of a single integration process, it is seen that a fairly accurate knowledge of position is available for the first five or six seconds. However, for longer elapsed times following the angular input rotation, this signal gradually disappears, leaving no recollection of rotation.

If one assumes that the knowledge of such rotation can be retained by a subject in the absence of all cues external to the semicircular canal signal, then a second integration process would be required.

Figure 16 shows the result of the double integration process on the semicircular canal response $\theta_e(t)$ with the assumption that this response represents that of an accelerometer with scale factor $\frac{1}{P}$. The actual input displacement function $\frac{\theta_i(t)}{A}$ is reproduced for comparison. It can be seen that as time progresses, the doubly integrated response function approaches the input function assymptotically, thus providing long term recall of the actual displacement.

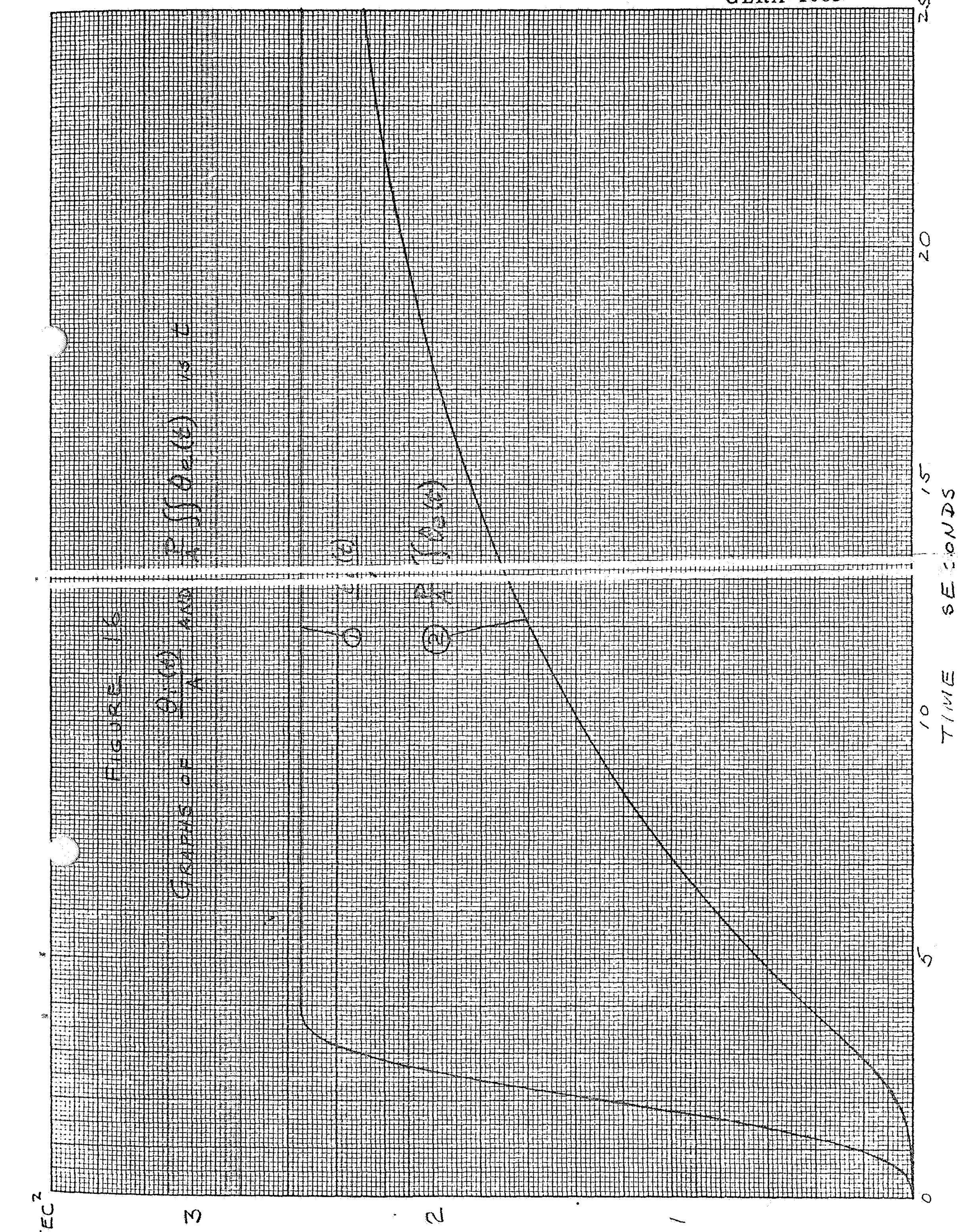

FIGURE 16

GRAPHS OF $\frac{\theta_i(t)}{A}$ AND $\frac{P}{A}\iint \theta_e(t)$ vs t

It may be of interest to consider what operations would be required on the semicircular canal response signals in order to generate a signal which would represent exactly any motion to which the canal is subjected. This signal, designated by $\theta_c(t)$, would be identical to the input motion $\theta_i(t)$. Thus,

$$\theta_c(t) = \theta_i(t) \tag{140}$$

and

$$\theta_c(s) = \theta_i(s) \ . \tag{141}$$

But, from Equation (4), the canal response was shown to be

$$\theta_e(s) = \frac{s^2}{s^2 + Ls + P}\ \theta_i(s) \ . \tag{142}$$

Consequently, for Equation (141) to be satisfied, the following relation must exist.

$$\theta_c(s) = \frac{s^2 + Ls + P}{s^2}\ \theta_e(s) \ , \tag{143}$$

or

$$\theta_c(s) = \quad 1 + \frac{L}{s} + \frac{P}{s^2} \quad \theta_e(s) \ . \tag{144}$$

Since a $\frac{1}{s}$ term represents a single integration process and a $\frac{1}{s^2}$ term represents a double integration process, it is seen from Equation (144) that an accurate computed position signal θ_c can be obtained by summing the following three signals.

a. The endolymph displacement signal θ_e with a multiplier of unity

b. The single integral of θ_e with a multiplier equal to the canal constant L

c. The double integral of θ_e with a multiplier equal to the canal constant P.

Considerable experimental evidence exists indicating that no signal such as θ_c appears to be generated. However, it is conceivable that all three of the above signals may be generated with perhaps multipliers other than those mentioned above, and that different combinations of such signals with perhaps the derivative of θ_e may be used in the various control and information functions using the semicircular canal as a basic angular motion-sensing device.

REFERENCES

1. Brown, J H, and Crampton, G H. Quantification of the human nystagmic response to angular acceleration. Prediction formulae and nomograph. Acta oto-laryng. 58, 555-564, 1964.

2. van Egmond, A A J, Groen, J J, and Jongkees, L B W. The mechanics of the semicircular canal. J Physiol. 110, 1-17, 1949.

3. Ek, J, Jongkees, L B W, and Klign, J A J. On the effect of continuous acceleration. Acta oto-laryng. 51, 416-419, 1960.

4. Gardner, M F, and Barnes, J L. Transients in Linear Systems, Vol I. John Wiley and Sons, Inc.

5. Groen, J J, Lowenstein, O, and Vendrik, A J H. The mechanical analysis of the responses from the end-organs of the horizontal semicircular canal in the isolated elasmobranch labyrinth. J Physiol. 117, 329-240. 1952.

6. Groen, J J. Problems of the semicircular canal from a mechanico-physiological point of view. Acta oto-laryng. Suppl. 163, 1960.

7. Guedry, F E. Habituation to complex vestibular stimulation in man: Transfer retention of effects from twelve days of rotation at 10 rpm. USN SAM/NASA Joint Report No. 109, Project MR005.13-6001, 26 Mar 1965.

8. Hixson, W C, and Niven, J I. Application of the system transfer function concept to a mathematical description of the labyrinth: I. Steady-state nystagmus response to semicircular canal stimulation by angular acceleration. USN SAM/NASA Joint Report No. 57, Project MR005.13-6001, 10 Mar 1961.

9. Hixson, W C, and Niven, J I. Frequency response of the human semicircular canals. II. Nystagmus phase shift as a measure of non-linearities. USN SAM/NASA Joint Report No. 73, Project MR005.13-6001, 26 Jul 1962.

10. Jones, G M and Milsum, J H. Spatial and dynamic aspects of visual fixation. IEEE Transactions and Bio-Medical Engineering, 54-62, 1965.

11. Mayne, R. The constants of the semicircular canal differential equation. Goodyear Aerospace Corp Report GERA-1083, 21 Oct 1965.

12. Mayne, R. The dynamic characteristics of the semicircular canals. J Comp Physiol Psychol, 43, No. 4, August 1950.

13. Mayne, R. The functional parameters of the semicircular canals. Goodyear Aerospace Corp Report GERA-1056, 23 Aug 1965.

14. Mayne, R. Nystagmus and body control for inputs of repeated patterns. Goodyear Aerospace Corp Report GERA-1080, 6 Oct 1965.

15. Niven, J I, and Hixson, W C. Frequency response of the human semicircular canals: I. Steady-state ocular nystagmus response to high-level sinusoidal angular rotations. USN SAM/NASA Joint Report No. 58, Project MR005.13-6001, 10 Mar 1961.

16. Steinhausen, W von. Uber die Beobachtung der Cupula in den Bogengangsampullen des Labyrinths des Lebenden Hechts. Pflug. Arch Ges Physiol, 232, 1933.

APPENDIX

The purpose of this appendix is to present some of the basic concepts in Laplace transformation theory which have been used in the various derivations presented in this report.

1. Real Differentiation (Theorem) (Ref. 4, p 127)

If the function f(t) and its derivative $\frac{df(t)}{dt}$ are Laplace transformable, and if f(t) has the Laplace transform F(s), then

$$\text{Laplace transform of } \frac{df(t)}{dt} = s\,F(s) - f(o^+) \tag{1A}$$

where $f(o^+)$ is the value of the function f(t) as the value of t is reduced from positive values down to zero.

Also,

$$\text{Laplace transform of } \frac{d^2 f(t)}{dt^2} = s^2 F(s) - sf(o^+) - f^1(o^+) \tag{2A}$$

where f^1 is the first derivative of f.

2. Real Integration (Theorem) (Ref 4, p 129)

If the function f(t) is Laplace transformable and has the Laplace transform F(s), its integral

$$f^{(-1)}(t) = \int f(t)\,dt = \int_o^t f(t)\,dt + f^{(-1)}(o^+) \tag{3A}$$

is likewise Laplace transformable, and

$$\text{Laplace transform of } \int f(t)\,dt = \frac{F(s)}{s} + \frac{f^{(-1)}(o^+)}{s} . \tag{4A}$$

Also,

$$\text{Laplace transform of } f^{(-2)}(t) = \frac{F(s)}{s^2} + \frac{f^{(-1)}(o^+)}{s^2} + \frac{f^{(-2)}(o^+)}{s} \qquad (5A)$$

where $f^{(-2)}$ is the double integral of f.

3. Partial Fraction Expansion (Ref 4, p 153)

Given a function of s such as

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_1 s + a_o}{s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_o} \qquad (6A)$$

in which the a's and b's are real constants, and p and q are positive integers, then, when $p < q$ and s_1, s_2, s_3, ... s_q are all different roots (no two roots equal) of $B(s) = 0$, then F(s) can be written as a sum of partial fractions as follows

$$F(s) = \frac{A(s)}{B(s)} = \frac{K_1}{s - s_1} + \frac{K_2}{s - s_2} + \ldots + \frac{K_k}{s - s_k} + \ldots + \frac{K_q}{s - s_q} \qquad (7A)$$

where the constants K_k, are evaluated by

$$K_k = \left[\frac{(s - s_k)\, A(s)}{B(s)}\right]_{s = s_k} . \qquad (8A)$$

4. Short Table of Laplace Transform Pairs

$F(s)$	$f(t)$ for $o \leq t$
$\dfrac{1}{s}$	1 or $u(t)$
$\dfrac{1}{s+\alpha}$	$e^{-\alpha t}$
$\dfrac{\beta}{s^2+\beta^2} = \dfrac{\beta}{(s+j\beta)(s-j\beta)}$	$\sin \beta t$
$\dfrac{s}{s^2+\beta^2} = \dfrac{s}{(s+j\beta)(s-j\beta)}$	$\cos \beta t$
$\dfrac{1}{s^2}$	t